OPTIMAL CONTROL

(6)

$$\text{Max} \int_{t_0}^{t_1} u(c) e^{-rt}\, dt$$ – objective functional

$$\text{ST} \quad C \le \cancel{f(k)}\ f(K) - \dot{K}$$ (continuum)

many constraints.

"Lagrangian" "$\mathcal{L}$" with multipliers $\mu(t)$

$$= \int_{t_0}^{t_1} u(c) e^{-rt}\, dt - \int_{t_0}^{t_1} \mu(t)\left[c(t) - f(K(t)) + \dot{K}(t)\right] dt$$

integrate by parts

$$-\int_{t_0}^{t_1} \mu(t)\, \dot{K}\, dt = -\Big|_{t_0}^{t_1} \mu(t) K(t) + \int_{t_0}^{t_1} \dot{\mu}(t) K(t)\, dt$$

$$\mathcal{L} = \int_{t_0}^{t_1} \left[u(c) e^{-rt} + \dot{\mu} K - \mu c + \mu f(K)\right] dt - \Big|_{t_0}^{t_1} \mu(t) K(t).$$

max integrand for each t.

$$u'(c) e^{-rt} = \mu. \qquad \dot{\mu} = -\mu f'(K). \qquad \dot{K} = f(K) - c.$$

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Maximizing the <u>RIGHT</u> (right shadow prices).

Lagrangian w.r.t. choice variable is <u>sufficient</u> for a constrained global max,

but <u>not</u> necessary.

$$\mathcal{L}_\mu(x) = f(x) - \mu^T[g(x) - a]$$

$$\max_x f(x) \text{ st } g(x) \leqq a.$$

$$f(x) - f(x^*) = \mathcal{L}_\mu(x) + \mu^T[g(x)-a] - \mathcal{L}_\mu(x^*) - \mu^T[g(x^*)-a]$$

$\boxed{\mathcal{L}_\mu(x) \leq \mathcal{L}_\mu(x^*)}$ x^* is a max.

$$\leqq \mu^T[g(x) - g(x^*)] \leq \mu^T[a - g(x^*)] = 0$$

$\mu \geqq 0$

$\mu \geqq 0, g(x) \leqq a$ (comp.).

Comp. slackness

$$\mu_i > 0 \Longrightarrow g_i(x^*) = a_i$$

$$g_i(x^*) < a_i \Longrightarrow \mu_i = 0.$$

$$\mu_i[g_i(x^*) - a_i] = 0$$

$$\max_{x(t),\,u(t)} \int_{t_0}^{t_1} f(t, x(t), u(t))\,dt$$ ← objective functional

$x(t)$: state, $u(t)$: control.

initial state $x(t_0) = x_0$, given.

($x(t_1)$ is free).

$(\leq)$

$$\dot{x} = g(t, x, u).$$ dynamic constraint.

costate variable (vector)

$$\mathcal{L} = \int_{t_0}^{t_1} f(t,x,u)\,dt - \int_{t_0}^{t_1} \mu\,[\dot{x} - g(t,x,u)]\,dt$$

$$= \int_{t_0}^{t_1} \left[\underline{f(t,x,u)} + \dot{\mu} x + \underline{\mu\, g(t,x,u)}\right] dt$$

$$- \Big|_{t_0}^{t_1} \mu x. = \int_{t_0}^{t_1} (H + \dot{\mu} x)\,dt + \dots$$

$\mu \dot{x} \to$

Hamiltonian

$$H(t, x, u, \mu) = f(t,x,u) + \mu\, g(t,x,u).$$

co-state.

Then u^* $\max_u H(t, x, u, \mu)$ (all t).

x: state

MAX PRINCIPLE.

$$\boxed{\dot{\mu} = -H'_x(t, x, u, \mu).}$$

partial gradient

$$\mu(t_1) = 0.$$

transversality.

(FOC for max $H + \dot{\mu} x$)

<u>REMARK.</u>

$$\dot{\mu} = -H'_x$$
$$= -f'_x \overset{-}{\cancel{+}} \mu g'_x$$

is the first-order condition for maximizing the integrand

$$f + \dot{\mu} x + \mu g \quad (\text{in } L)$$

w.r.t. x.

WARNING. Max w.r.t. x is sufficient, but only stationarity is necessary.

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"Macro" control problem:

$$\min \int_0^T (x^2 + cu^2)\,dt$$

ST $\dot{x} = u$, $x(0) = x_0$, $x(T)$ free.

$$H = -x^2 - cu^2 + pu$$

u^*, $H'_u = 0$, $-2cu + p = 0$

$$u^* = p/2c.$$

$$\dot{p} = -H'_x = 2x. \quad (*)$$

$$\dot{x} = p/2c \quad (**)$$

$$\ddot{p} = 2\dot{x} \cancel{= \dot{p}/(2c = \dot{p}/c}$$

$$= \cancel{p/2c}\, p/c$$

$$c\ddot{p} - p = 0$$

$$p = Ae^{\lambda_1 t} + Be^{\lambda_2 t}.$$

$$c\lambda^2 - 1 = 0.$$

$$\lambda_1, \lambda_2 = \pm c^{-\frac{1}{2}}, \quad r = 1/\sqrt{c}.$$

$$p = Ae^{rt} + Be^{-rt}.$$

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$$\mu = Ae^{rt} + Be^{-rt}, \quad r = 1/\sqrt{c}.$$

$$\mu(T) = 0, \quad x(0) = x_0,$$

$$\dot{\mu} = rAe^{rt} - rBe^{-rt}$$

$$\cancel{\dot{x} = \mu/2c.}$$

$$\cancel{Ae^{rT} + Be^{-rT} = 0.}$$

$$\cancel{\dot{x} = \frac{A}{2c}e^{rt} + \frac{B}{2c}e^{-rt}}$$

$$\cancel{x(t) = x_0 + \frac{1}{2c}\int_0^t \left(Ae^{rs} + Be^{-rs}\right) ds.}$$

$$\cancel{= x_0 + \frac{1}{2cr}\Big[\left(Ae^{rs} - Be^{-rs}\right)\Big]_0^t}$$

$$\cancel{= x_0 + \frac{1}{2cr}\left[\left(Ae^{rt} - Be^{-rt}\right) - (A-B)\right]}$$

$$\dot{\mu}(0) = 2x_0 = r(A-B).$$

$$Ae^{rT} + Be^{-rT} = 0.$$

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$$\mu(t) = \frac{2x_0\left[e^{-r(T-t)} - e^{r(T-t)}\right]}{r\left(e^{rT} + e^{-rT}\right)}$$

$$x^*(t) = x_0 \frac{e^{r(T-t)} + e^{-r(T-t)}}{e^{rT} + e^{-rT}}$$

What happens as $T \to \infty$? 12

Multiply by e^{-rT}.

$$p(t) = -2x_0 e^{-rt}/r \to 0$$

$$x^*(t) = x_0 e^{-rt} \to 0 \text{ as } t \to \infty.$$

(x, u) to max $H(t, x, u, \mu) + \dot{\mu} x$ 14 ~~13~~ 14 13

<u>Sufficient conditions</u> for a stationary pt. (x^*, u^*) to be a global max.

1) H is concave in (x, u),
~~stati~~ (x^*, u^*) stationary pt.
$\dot{\mu} = -H'_x$ ⟸ FOC.

Thm 9.7.1 Mangasarian.

2) Define $\hat{H}(t, x, \mu) := \max_u H(t, x, u, \mu)$,
and suppose $\hat{H}$ is concave in x.
$\dot{\mu} = -H'_x$ as FOC for
$$\max_x \hat{H}(t, x, \mu) + \dot{\mu} x.$$

Thm 9.7.2 Arrow.

$\dot{\mu} = -H'_x = -\hat{H}'_x$ by envelope theorem.

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$$\max \int_0^T f(t,x,u)\,dt$$

ST $\dot{x} \leqq g(t,x,u)$ (all t).

$x(0) \leqq x_0$, $x(T) \geqq x_T$.

~~fixed end point~~.

$$H(t,x,u,p) = f(t,x,u) + p\,g(t,x,u)$$

$(\dot{x})$

Suppose (x^*, u^*) max $H(t,x,u,p) + \dot{p}x$

for all t, with $p(t) \geqq 0$, $\dot{x}^* \leqq g(t,x^*,u^*)$ (comp.)

and $p(0) \geqq 0$, $x(0) \leqq x_0$ (comp.)

$p(T) \geqq 0$, $x(T) \geqq x_T$ (comp.)

Consider an alternative feasible path $x(t)$ $u(t)$ satisfying the inequality constraints.

Then

$$D := \int_0^T [f(t,x,u) - f(t,x^*,u^*)]\,dt$$

$$\leqq \int_0^T [-p\,g(t,x,u) + p\,g(t,x^*,u^*)]\,dt.$$

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$$D = \int_0^T (f - f^*)\, dt$$

where $f = H - \mu g$, $f^* = H^* - \mu g^*$

$$D = \int_0^T (H - \mu g - H^* + \mu g^*)\, dt$$

$$\leq \int_0^T (-\mu g + \mu g^*)\, dt$$ because $H \leq H^*$.

$$\leq \int_0^T (-\mu \dot{x} \cancel{dt} + \mu \dot{x}^*)\, dt$$

because $\mu \geq 0$, $\dot{x} \leq g(t, x, u)$,

and because $\mu \geq 0$, $\dot{x}^* \leq g(t, x^*, u^*)$ (comp)

$$= \Big|_0^T -\mu x + \mu x^*$$

$$\cancel{\leq \mu(T)[x^*(T) - x(T)] - \mu(0)[x^*(0) - x(0)]}$$

$$= \mu(T)[x^*(T) - x(T)] - \mu(0)[x^*(0) - x(0)]$$

$$- \mu(0)[x^*(0) - x(0)] - \mu(0)x_0 + \mu(0)x_0$$

$$\leq \mu(T)x_T - \mu(T)x_T - \mu(0)x_0 + \mu(0)x_0$$

$$= 0$$

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max $\int_0^\infty f(t,x,u)\,dt$

ST $\dot{x} = g(t,x,u)$.

$x(0) = x_0$, $x(t) \geq \underline{x}$ (all t).

$H(t,x,u,p) = f(t,x,u) + p\,g(t,x,u)$.

(x^*, u^*) max $H(t,x,u,p) + \dot{p}x$. $\quad p \geq 0$.

~~$p(0)$, $x(0) \leq x_0$ Comp. Slack.~~

$$D(T) = \int_0^T [f(t,x,u) - f(t,x^*,u^*)]\,dt$$

$$\leq \int_0^T p(x^* - x) = p(T)[x^*(T) - x(T)]$$

~~because $p(0)x^*(0) = p(0)x_0$~~

$$\leq p(T)[x^*(T) - \underline{x}] \to 0 \text{ as } T \to \infty.$$

Sufficient Transversality condition. we desire

$\Rightarrow \int_0^\infty f \leq \int_0^\infty f^*$. $\quad p(T)x^*(T) \to 0$ as $T \to \infty$.

MALINVAUD.

$p(T) \to 0$, $x^*(T)$ is bounded,

$p(T) \geq 0$ $\underline{x} \geq 0$.

CONSTANT exponential discounting

$$\tilde{f}(t, x, u) \equiv e^{-rt} f(x, u).$$

$r > 0$ constant.

$$H = \tilde{f} + p g$$

$$\dot{x} = g(t, x, u).$$

$$= e^{-rt}[f(x, u) + q g]$$

$$p = e^{-rt} q, \quad q = e^{rt} p.$$

$$H = e^{-rt} \tilde{H}, \quad \tilde{H} = f + q g$$

~~PRESENT~~ CURRENT VALUE HAMILTONIAN.

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$$H(t, x, u, p) = e^{-rt} f(x, u) + p\, g(x, u)$$

ind. of t.

$$= e^{-rt} \hat{H}(x, u, q)$$

~~present~~ current value Hamiltonian

$$\boxed{\text{where } \hat{H}(x, u, q) = f(x, u) + q\, g(x, u)}$$

$$q e^{-rt} = p.$$

q = ~~present~~ current value costate variable.

$$\dot{p} = -H'_x = -e^{-rt} \hat{H}'_x$$

$$\dot{p} = \dot{q} e^{-rt} - rq e^{-rt}$$

$$\dot{q} = rq + e^{rt} \dot{p}$$

$$= rq - \hat{H}'_x.$$

$$\dot{q} - rq = -\hat{H}'_x.$$

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???

$$H(t, x, u, \mu)$$
$$= f(t, x, u) + \mu\, g(t, \overset{\dot{x}}{x}, u)$$

$$H(t, x, u, \mu)$$
$$= e^{-rt}\left[f(x, u) + \mu\, g(x, u)\right]$$

$$\tilde{H}(t, x, u, q) = e^{-rt} f(x, u) + q\, g(x, u).$$

$$q = e^{-rt} \mu.$$

$$\dot{q} = -rq + e^{-rt}\dot{\mu} = -\tilde{H}'_x$$

$$\cancel{\dot{\mu} = e^{rt}(\dot{q}}$$

$$\cancel{\dot{\mu} = -e^{rt}\left[H'_x + \dot{q} - rq\right]}$$

$$\dot{\mu} = e^{rt}(\dot{q} + rq) = e^{rt}(-\tilde{H}'_x + rq)$$

$$\dot{\mu} = r\mu - H'_x$$

$$\text{Max} \int_0^\infty u(c)\, e^{-rt}\, dt.$$

$$\text{ST} \quad \dot{k} = ak - bk^2 - C$$

$$u(c) = \frac{c^{1-\varepsilon}}{1-\varepsilon}, \quad u'(c) = c^{-\varepsilon}$$

$$\varepsilon > 0.$$

isoelastic

constant relative rate of fluctuation aversion

$(\varepsilon \neq 1)$. $\quad \ln c$. if $\varepsilon = 1$

$$a > r > 0$$

Current value Hamiltonian.

$$\tilde{H} = u(c) + \lambda\left(ak - bk^2 - C\right)$$

$$u'(c) = \lambda, \quad c = \lambda^{-1/\varepsilon}$$

$$\dot{\lambda} = r\lambda - \tilde{H}'_k$$

$$= r\lambda - \lambda a + 2\lambda b k$$

$$\boxed{\begin{aligned} \dot{\lambda} &= 2b\lambda\left(k - \frac{a-r}{2b}\right) \\ \dot{k} &= ak - bk^2 - \lambda^{-1/\varepsilon} \end{aligned}}$$

$$\dot{\lambda} = 2b\lambda\left(k - \frac{a-r}{2b}\right)$$

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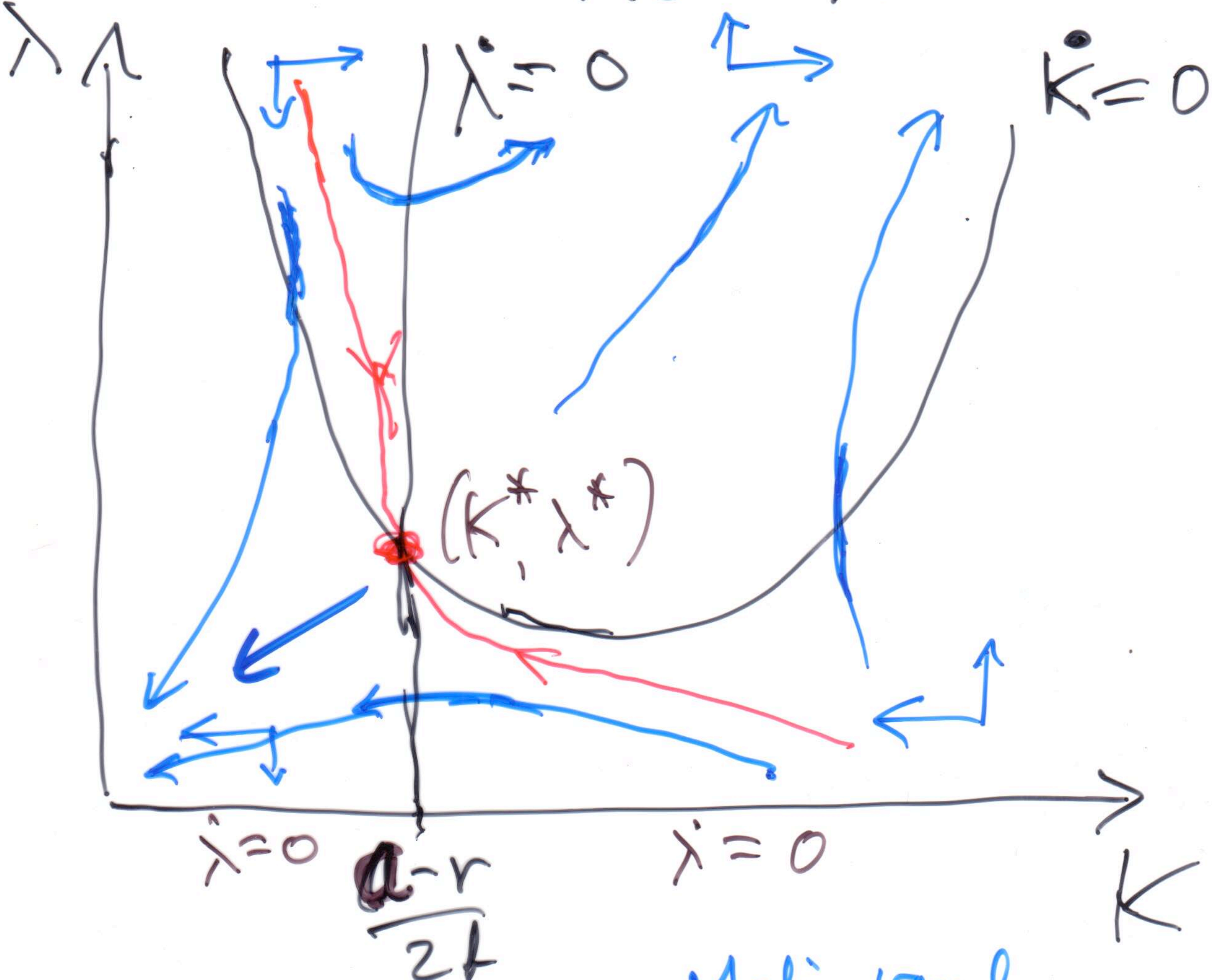
$$\dot{k} = ak - bk^2 - \lambda^{-1/\varepsilon}$$



Malinvaud transversality.

(k^*, λ^*) $e^{-rt}\lambda^* k^*$

$e^{-rt}\lambda k \longrightarrow 0.$

$c = \lambda^{-1/\varepsilon}, \; \lambda = c^{-\varepsilon}.$

Red path is optimal if ~~≥~~ $r > 0$

unclear if $r = 0$.