The Efficiency Theorems and Market Failure

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1.   Introduction

1.1.   Consumer sovereignty

The general equilibrium analysis of perfectly competitive markets plays a central role in most attempts by positive economics to describe what happens in a market economy. It is usually admitted that there may be barriers to competition, that markets may be incomplete, and information may be lacking. Nevertheless, as a theoretical ideal which may approximate reality, general equilibrium analysis is a widely used tool.

In normative economics, however — often called "welfare economics" because of its claim to be about how to enhance well-being or welfare — general equilibrium analysis has been if anything even more important than in positive economics. The reason for this is the striking relationship between, on the one hand, allocations that emerge from complete markets in perfectly competitive equilibrium, and on the other hand, allocations satisfying the normative property of Pareto efficiency. The latter are defined as allocations which at least meet the following necessary condition for normative acceptability: it is impossible to reform the economic system in a way that makes any consumer better off without at the same time making some other consumer worse off. As I say, this seems like a necessary condition for normative acceptability because, if it were not met, one could re-design the economic system so that at least one consumer gains without anybody losing. It is surely not a sufficient condition, however, because Pareto efficiency is compatible with extremely unjust distributions of consumption goods and leisure. For example, suppose that one dictator is served by a group of slaves, and consumes everything except the minimum needed to keep these slaves alive. Such an arrangement will be Pareto efficient if there is no way in which the dictator could possibly be made better off, and if no slave could gain unless another

loses. Indeed, slavery can easily be compatible with Pareto efficiency (Bergstrom, 1971). So can starvation, if the only way to relieve starvation is by making some of those who would survive anyway worse off (Coles and Hammond, 1995).

Even as a necessary condition for ethical acceptability, the criterion of Pareto efficiency is far from unquestionable. Indeed, it presumes a form of "welfarism" which Sen (1982, 1987) has often criticized. A response to Sen might be to re-define an individual i's welfare as that aim which it is ethically appropriate to pursue when only individual i is affected by the decisions being considered. Then, however, another crucial assumption becomes open to question — namely, that of "consumer sovereignty." This identifies each consumer i's welfare with a complete preference ordering that is meant to explain i's demands within the market system. Market outcomes can hardly be expected to be ethically satisfactory if consumers choose things they should not want. Of course one can argue — as many ethical theorists do — that it is nearly always right to let consumers have what they want, partly because they are often the best judges of what is good for them, but also because freedom is something to value for its own sake.

Yet these really are *assumptions* — even *ethical value judgements* — which should not be allowed to slip by without any comment at all. Indeed, most governments suppress trade in narcotic drugs and make school attendance compulsory for children within a certain age range precisely because they do not accept that consumer sovereignty is appropriate in all cases. So important ethical issues are indeed at stake.

Nevertheless, this chapter is not really about ethics as such, but rather about the circumstances in which markets can and cannot produce allocations that are normatively acceptable. To limit the ground that has to be covered, from now on I shall consider only the case in which consumer preferences are treated as sovereign. This is virtually equivalent to conceding that Pareto efficiency is a necessary condition for acceptability. This means that it is right to focus attention on the "Pareto frontier" of efficient allocations. As remarked above, however, not all Pareto efficient allocations are ethically acceptable to most people, but only those which avoid the extremes of poverty and of inequality in the distribution of wealth.

1.2. Two Efficiency Theorems

Following Arrow's (1951) pioneering work in particular, the following discussion will distinguish between two different results relating allocations that emerge from equilibrium in complete competitive markets to those that are Pareto efficient. The *first efficiency theorem* says that such market allocations are always "weakly" Pareto efficient, at least, in the sense that no other feasible allocation can make all consumers better off simultaneously. Moreover, if consumer's preferences satisfy a mild condition of "local non-satiation" which will be explained in Section 2 below, then market allocations will be (fully) Pareto efficient. This is a weak result insofar as there is no guarantee of anything like distributive justice in the market allocation, since markets by themselves cannot undo any injustice in the initial distribution of resources, skills, property, etc. Yet it is also a very strong result because it relies on such extraordinarily weak assumptions.

The *second efficiency theorem*, by contrast, is a form of converse to the first theorem. That first theorem shows how having complete competitive markets is sufficient for Pareto efficiency. The second theorem claims that the same condition is necessary for Pareto efficiency — that any particular Pareto efficient allocation can be supported by setting up complete competitive markets and having them reach equilibrium. But there are some very important qualifications to this claim.

First, unless wealth is suitably redistributed, markets will generally reach an entirely different equilibrium from any particular Pareto efficient allocation that may be the target. In "smooth" economies such as those considered in Chapter 5(??) of this volume, for a fixed distribution of wealth there will typically be at most a finite set of different possible equilibrium allocations. In determining the set of all Pareto efficient allocations, however, there are typically $n - 1$ degrees of freedom, where n is the number of individuals, and even an $(n - 1)$-dimensional manifold of such allocations. For example, when $n = 2$ as in an Edgeworth box economy, there is usually a one-dimensional curve of Pareto efficient allocations. So, to repeat, there has to be lump-sum redistribution of wealth before markets can reach a particular Pareto efficient allocation.

Second, even if in principle the distribution of wealth allows a desired Pareto efficient allocation to be reached as an equilibrium, there may be other equilibria, including some that are much less desirable. Worse, the desired equilibrium may be unstable, or at least

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less stable than an undesirable one. These issues are discussed in Bryant (1994) — see also Samuelson (1974).

The other major qualifications arise because, even if appropriate lump-sum redistribution occurs, it is still not generally possible for complete competitive markets to achieve equilibrium at a particular Pareto efficient allocation. After all, some assumptions are needed to ensure that competitive equilibrium exists — e.g., continuous convex preferences, a closed convex production set, etc. The first efficiency theorem needed no such assumptions because its hypothesis was that an equilibrium had already been reached, implying that such an equilibrium must exist. The second efficiency theorem, by contrast, relies on extra assumptions which guarantee that there are equilibrium prices at which the given Pareto efficient allocation will be a complete competitive market equilibrium, for a suitable distribution of wealth.

In the end, it is possible to combine the two theorems into one single characterization result. This says that, allowing for all possible systems of lump-sum wealth redistribution, the entire set of competitive equilibrium allocations coincides with the entire set of Pareto efficient allocations. There are difficulties, however. Obviously, the stricter conditions of the second efficiency theorem have to be assumed. Even then, as the later sections show, there can still be difficulties with some "oligarchic" allocations where the distribution of wealth is at an extreme. So the correspondence between market and efficient allocations is rarely exact. Even when it is, however, the conditions for the first efficiency theorem to hold are so much weaker than those for the second that it is surely worth treating them as two separate results.

1.3. Outline of Chapter

In the following pages, Section 2 sets out the notation that will be used to describe consumers and their demands, as well as the assumptions that will be made about their feasible sets and preferences. Section 3 does the same for producers. Thereafter, Section 4 considers which allocations are feasible and which among the feasible allocations are weakly or fully Pareto efficient. Section 5 considers the relevant notions of market equilibrium. In fact, it is useful to consider several different notions. Not only must an ordinary Walrasian equilibrium be considered but, in order to allow markets to reach *any* point of the Pareto

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frontier, it is important to consider Walrasian equilibrium with price dependent lump-sum redistribution of wealth. It is also helpful to distinguish between ordinary "uncompensated" demands, which arise when each consumer's wealth is treated as exogenous and preferences are maximized, from "compensated" demands. The latter arise when each consumer's wealth changes so as to compensate for price changes in a way that maintains their "standard of living," and also minimizes expenditure over the "upper contour set" of points that are weakly preferred to some *status quo* allocation. Section 5 recalls Arrow's "exceptional case" and Debreu's example of "lexicographic preferences" in order to illustrate the difference between compensated and uncompensated demands. It finishes with the "Cheaper Point" theorem" that provides a sufficient condition for the two kinds of demands to be identical.

After these essential preliminaries, the standard results relating market equilibrium to Pareto efficient allocations can be presented. Section 6 begins with the first efficiency theorem, stating that a competitive allocation is at least weakly Pareto efficient. Moreover, under local non-satiation or some other extra condition guaranteeing that the competitive allocation is also compensated competitive, it will also be fully efficient. An example shows, however, that without such an extra assumption, a competitive allocation may not be fully efficient, even though it must always be weakly efficient.

Section 7 turns to the more difficult second efficiency theorem, which is an incomplete converse to the first efficiency theorem. The claim of the second theorem, recall, is that any Pareto efficient allocation can be achieved by setting up complete competitive markets with a suitable distribution of wealth, and steering them toward the appropriate equilibrium. The second theorem, however, is only true under several additional assumptions. Whereas the first efficiency theorem never needs more than local non-satiation even to conclude that competitive allocations are fully Pareto efficient. In fact Section 7 only gives sufficient conditions for a Pareto efficient allocation to be *compensated* competitive. These conditions are that the aggregate production set be convex, and that consumers have convex and locally non-satiated complete preference orderings.

In order to go from compensated to uncompensated equilibria, Section 8 introduces three additional assumptions. The first is continuity of individual preferences, which will play a crucial role in proving the "Cheaper Point" theorem of Section 5, showing when a compensated equilibrium would also be uncompensated. Two further assumptions, however,

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are needed to rule out examples like Arrow's exceptional case. Of these, the first is that all commodities are "relevant" in the sense that the directions in which one can move from one feasible allocation to another span the whole of the commodity space, and not just some limited subspace which excludes some "irrelevant" commodities. The second, which seems new to the literature (see also Hammond, 1992), is a "non-oligarchy" assumption. This rules out allocations which concentrate wealth in the hands of an "oligarchy" to such an extreme that there is no way for all members of the oligarchy to become better off, even if they were allowed to use as they pleased all the resources of those consumers who are excluded from the oligarchy. Finally, then, under the assumptions that the aggregate production set is convex, that preferences are complete, continuous, convex and locally non-satiated, and that all commodities are relevant, it is proved that any non-oligarchic Pareto efficient allocation can be achieved as a competitive allocation for a suitable lump-sum redistribution of initial wealth.

It is commonly believed that public goods and externalities give rise to "market failures," in the sense that they prevent even perfectly competitive markets from being used to achieve Pareto efficient allocations. Sections 9 and 10 address this issue, and come to a more subtle conclusion. It turns out that both private goods and externalities can be treated in a common theoretical framework with a "public environment" which is affected by the decisions of individual consumers and firms to "create externalities" or to contribute the private resources needed to produce public goods. Furthermore, there is an equivalent private good economy in which externalities (or the rights or duties to create them) are traded, along with not only ordinary private goods and services, but also "individualized" copies of the public environment. In principle, the latter allow all consumers and producers to choose and pay for their own separate versions of the public environment. Ultimately, though, the individualized "Lindahl prices" used to allocate all these versions will clear the market by encouraging everybody to demand one and the same environment in equilibrium. For each different aspect of the environment, such as the quantity of one particular form of air pollution in one particular area, this Lindahl price will vary from consumer to consumer, and also from producer to producer, in order to reflect the marginal benefit or damage that each consumer and producer experiences from that aspect. Indeed, even the signs of individualized Lindahl prices for the same aspect of the environment may differ between different agents in the economy.

Moreover, in the case of a privately created externality, individuals' contributions to the corresponding aspect of the environment, such as the amount of pollution they cause, will be charged for or subsidized, as appropriate, by "Pigou prices" — which are, in effect, taxes or subsidies on the creation of each kind of externality. Unlike Lindahl prices, however, these Pigou prices will be the same for all agents. Indeed, for the right to create a negative externality, everybody will be expected to pay a price per unit of externality that is equal to the total of the marginal damages inflicted on all agents in the economy, as reflected in the sum of the appropriate Lindahl prices. In the case of a positive externality, this Pigou price will be negative, and represent the payment to the agent for assuming the duty to create a certain amount of that externality.

This combination of Lindahl and Pigou prices will be called a "Lindahl-Pigou pricing scheme." It allows all the earlier results on private good economies to go through without any alteration except, of course, to their interpretation. There is a serious snag, however, which concerns the plausibility of the usual assumptions — especially the convexity assumptions that are usually required to make the second efficiency theorem true. For, as Starrett (1972) pointed out, there is a clear sense in which negative externalities are always associated with "fundamental" non-convexities. This will be discussed in Section 11. So will some other obstacles to Pareto efficiency that prove more troublesome than just public goods and externalities on their own. Examples include physical transactions costs, as well as limited information.

The concluding Section 12 presents a brief summing up.

2. Consumers' Feasible Sets and Preferences

2.1. Commodities and net demands

Suppose that the different physical commodities or goods are labelled by the letter g, with g belonging to the finite set G. Different kinds of labour will be treated as particular goods as well. All goods are distinguished by time and by location, as necessary. Also, where there is uncertainty, goods may be distinguished by the commonly observable contingency or event which determines whether or not they should actually be delivered — see, for instance, Chapter 7 of Debreu (1959). The commodity space, therefore, is the finite-dimensional Euclidean space $\Re^G$.

A consumer's impact on the economy is described by the quantities of goods demanded and supplied. The distinction between demands and supplies for a consumer is unnecessarily cumbersome, however. For each good $g \in G$, the consumer's *net* demand for g is defined as the demand for g minus the supply of g. Demands and supplies are then distinguished by the sign of the net demands for the various goods. Where a consumer is a trader who is simultaneously both a demander and a supplier of the same good, it is the net demand which measures that consumer's impact on the rest of the economy. Accordingly, it will be enough to consider only each consumer's net demands in future.

Thus, each consumer will have a *net demand vector* x, which is some member of the commodity space $\Re^G$. The vector $x = (x_1, x_2, \ldots, x_n) = (x_g)_{g \in G}$ has components x_g ($g \in G$); each component x_g indicates the consumer's net demand for good g.

2.2. Feasible sets

The typical consumer's feasible set X is some closed subset of $\Re^G$. It is defined as the set of physically possible net demand vectors — i.e., x is a member of X if and only if the consumer has the capacity to make the net demands (and so provide any positive net supplies) which x represents. Note that the capacity of the economy to meet certain demands is something which the economist naturally takes as endogenous. So it is not reflected in this feasible set, which the economist takes as exogenous, and unaffected by the economy's ability or inability to meet the net demand vectors that make up the set.

Say that the consumer's feasible set *allows free disposal* if, whenever $x \in X$ and $x' \geqq x$, then $x' \in X$. For if $x \in X$ and $x' \geqq x$, then free disposal must indeed mean that x' is also feasible for the consumer, since the vector of quantities $x' - x \geqq 0$ can be freely disposed of in order to move from x to $x' = x + (x' - x)$ which is therefore feasible.[1]

In fact it will not be necessary to assume that X allows free disposal. However, the second efficiency theorem presented in Section 8 will rely on the assumption that each consumer has a *convex* feasible set X. In other words, whenever $x, x' \in X$ and whenever

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[1] The following notation will be used for vector inequalities in $\Re^G$:

(i) $x' \geqq x \iff \forall g \in G : x'_g \geq x_g$;

(ii) $x' > x \iff [x' \geqq x \quad \text{and} \quad x' \neq x]$;

(iii) $x' \gg x \iff \forall g \in G : x'_g > x_g$.

$\lambda, \mu \in \Re$ are two *convex weights* in the sense that they satisfy both $\lambda, \mu \geq 0$ and $\lambda + \mu = 1$, then the associated *convex combination* $\lambda x + \mu x'$ must be a member of X also.

2.3. Preferences

A consumer's preferences correspond to three binary relations on the set X. These are the *strict preference* relation P, the *indifference* relation I, and the *weak preference* relation R between pairs in X.

Some additional notation and terminology will also prove useful later on. First, the set $P(x) := \{\, x' \in X \mid x' \; P \; x \,\}$ will be called the *strict preference set* for x.[2] Second, the set $I(x) := \{\, x' \in X \mid x' \; I \; x \,\}$ will be called the *indifference set through* x; very often, as we shall see later, it collapses to an *indifference curve*. Third, the set $R(x) := \{\, x' \in X \mid x' \; R \; x \,\}$ will be called the *upper contour set* for x. Finally, the set $R^-(x) := \{\, x' \in X \mid x \; R \; x' \,\} = X \backslash P(x)$ will be called the *lower contour set* for x.

In the special case when the consumer's feasible set X satisfies free disposal, it is also plausible to assume that preferences are monotone, in at least one of the three different possible senses set out below. First, *weakly monotone* preferences satisfy the property that, whenever $x \in X$ and $x' \geqq x$, then $x' \; R \; x$. It can be interpreted as saying that no goods are undesirable. Second, preferences are said to be *monotone* if they satisfy this definition of weak monotonicity and if, in addition, whenever $x \in X$ and $x' \gg x$, then $x' \; P \; x$. This asserts that some arbitrarily small combination of goods is always desirable. Third, preferences are said to be *strictly monotone* if they are weakly monotone and if, in addition, whenever $x \in X$ and $x' > x$, then $x' \; P \; x$. Thus, even when the quantity of just one good increases, the consumer is better off, and so *all* goods are desirable in this last case.

Monotone preferences have some appeal when all goods are for private consumption because then a consumer is not often required to face large costs for the disposal of unwanted goods. Moreover, there is always some (luxury) good which remains desirable, no matter how well-off the consumer may be. For the public environment and externalities, however, free disposal will be a poor assumption. Accordingly, I shall not impose free disposal or the associated condition that preferences are monotone. These assumptions will be replaced with the following somewhat weaker condition.

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[2] The symbol := should be read as "(is) defined as equal to."

The consumer's preferences are *locally non-satiated* if, given any $x \in X$ and any neighbourhood N of x, there exists $x' \in N \cap X$ such that x' P x. Thus there are always arbitarily small changes away from x which the consumer prefers. When preferences are representable by a utility function, this is equivalent to the utility function having no local maximum (either weak or strict) in its domain X.

Local non-satiation obviously rules out "thick" indifference curves. Another way of expressing the requirement for local non-satiation is that $x \in$ cl $P(x)$ for every $x \in X$, where "cl" denotes the closure. Equivalently, for every $x \in X$, there must exist an infinite sequence $x_n \in P(x)$ $(n = 1, 2, \ldots)$ such that $x_n \to x$. This is because $x \notin P(x)$, and so one can have $x \in$ cl $P(x)$ if and only if every neighbourhood N of x contains points of $P(x)$ — i.e., iff there is local non-satiation at x.

Weak monotonicity allows indifference curves to be thick, and so does not imply local non-satiation. Monotonicity, however, does imply local non-satiation, because any neighbourhood N of a point $x \in X$ includes other points x' such that $x' \gg x$; then $x' \in X$ and x' P x because of monotonicity. Of course strict monotonicity, which trivially implies (ordinary) monotonicity, must imply local non-satiation *a fortiori*.

2.4.   Convexity

The consumer's preferences are said to be *convex* if:

 (i) the feasible set X is convex;

(ii) for every $x \in X$, the upper contour set $R(x)$ is convex.

The following important implication of preferences being convex will be used later in the proof of the second efficiency theorem:

PROPOSITION 2.1. *If a consumer has convex preferences, then for every $x \in X$ the strict preference set $P(x)$ is convex.*

PROOF: Suppose that $x^1, x^2 \in P(x)$ and that $x^0 = \lambda x^1 + \mu x^2$ is a convex combination. The preference relation is R is complete. Hence, it loses no generality to assume that the labels of the two points x^1 and x^2 have been chosen so that $x^1 \in R(x^2)$, as illustrated in Fig. 1. Because preferences are reflexive, $x^2 \in R(x^2)$. Therefore, because of convex preferences, it follows that x^0 R x^2. But x^2 P x by hypothesis, so x^0 P x by transitivity, as required. ∎
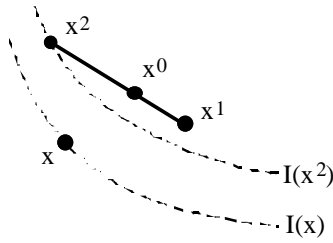
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Figure 1

2.5. Continuity

In addition to convexity of preferences, the following Sections 5 and 8 will also use the assumption that preferences are *continuous* in the sense that, for every $x \in X$, both the upper and lower contour sets $R(x)$ and $R^-(x)$ are closed. This implies that the union of these two sets, which is the entire feasible set X, and the intersection of these two sets, which is the indifference set $I(x)$, are also both closed sets. On the other hand, the preference set $P(x)$ is equal to the intersection of X with the open set $\Re^G \setminus R^-(x)$, and so must be open relative to X.

2.6. Many consumers

All the discussion above pertains to a typical individual consumer. It will be assumed that there is a finite set I of such consumers,[3] each indicated by a superscript $i \in I$. Thus x_g^i will denote consumer i's net trade for the specific commodity g, while $x^i = (x_g^i)_{g \in G}$ will denote consumer i's typical net trade vector, which should be a member of i's feasible set X^i. Moreover, i's three preference relations will be denoted by P^i, I^i, and R^i respectively.

A list of net demand vectors $\mathbf{x}^I = (x^i)_{i \in I}$, one for each consumer, will often be called a *distribution*.

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[3] Even though I is being used to denote both the set of consumers and an indifference relation, in practice there should be no confusion.

3. Producers

It will also be assumed that there are several different producers in the finite set J, indexed by the letter j. Superscripts will be used to denote different producers. Then y_g^j will denote the net output of good g by producer j — that is, output minus input. Just as it was enough to consider consumers' net demands, so is it enough to consider producers' net outputs — especially as, unless the producer wastes inputs or outputs, net outputs must be equal to net supplies. And y^j will denote the net output vector of producer j, whose components are y_g^j $(g \in G)$. Each producer j has technical production possibilities described by the production set Y^j.

A *production plan* is a complete list of net output vectors $\mathbf{y}^J = (y^j)_{j \in J}$, one for each producer $j \in J$, such that each individual net output vector satisfies $y^j \in Y^j$. In other words, it must be true that $\mathbf{y}^J \in \mathbf{Y}^J$ where $\mathbf{Y}^J$ denotes the Cartesian product $\prod_{j \in J} Y^j$ of all the firms' production sets. Effectively, such a production plan lays out a description of what every producer in the economy is doing. Actually, the term "plan" may be somewhat misleading, since the manner in which the economy arrives at a specific $\mathbf{y}^J \in \mathbf{Y}^J$ may be wholly unsystematic; there is no presumption that any kind of formal planning procedure is being used. Given the production plan $\mathbf{y}^J$, the corresponding aggregate net output vector is $y = \sum_{j \in J} y^j$.

The sets Y^j $(j \in J)$ describe what the producers of an economy can achieve separately, but it is usually more interesting to know what they can achieve collectively. If there are just two producers 1 and 2 who produce the net output vectors y^1 and y^2 separately, then their collective net output is described by the aggregate net output vector $y^1 + y^2$. To describe the possibilities of producers 1 and 2 acting together, it is therefore natural to define the *vector sum* of their two production sets Y^1 and Y^2 as

$$Y^1 + Y^2 := \{\, y \in \Re^G \mid \exists y^1 \in Y^1; \exists y^2 \in Y^2 : y = y^1 + y^2 \,\}.$$

Thus $Y^1 + Y^2$ is the set of all possible aggregate net output vectors y which can be obtained as the sum of any two vectors $y^1 \in Y^1$ and $y^2 \in Y^2$. Of course, this is precisely the set of aggregate net output vectors which the two firms 1 and 2 can produce together.

With a finite set J of firms, the aggregate production set Y is just the vector sum $\sum_j Y^j$ of all the production sets of the different producers in the economy, defined as

$$Y = \sum_{j \in J} Y^j = \{\, y \in \Re^G \mid \exists y^j \in Y^j \ (j \in J) : y = \sum_{j \in J} y^j \,\}.$$

4. Pareto Efficient Allocations

4.1. Feasible allocations

An allocation is a complete description of the impact that each agent has on the economy. It involves specifying each consumer's net demand vector, as well as each producer's net output vector. As long as the economy is closed and has no government, no public goods, and no externalities, that is all. Knowing what each consumer does and what each producer does is enough to know everything relevant about such an economy.

Formally, an *allocation* is:

(1) a distribution $\mathbf{x}^I \in \mathbf{X}^I := \prod_{i \in I} X^i$; and

(2) a production plan $\mathbf{y}^J \in \mathbf{Y}^Y := \prod_{j \in J} Y^j$; such that

(3) $\sum_{i \in I} x^i = \sum_{j \in J} y^j$.

The last vector equality is a "resource balance constraint." For each good $g \in G$, the total net supply is $\sum_j y_g^j$, and the total net demand is $\sum_i x_g^i$. The resource balance constraint ensures that the total net supply of each good is exactly enough to meet the total net demand. Notice, then, that an allocation has been defined so that it is always physically feasible. Indeed, (1) above ensures physical feasibility for each individual consumer $i \in I$, while (2) ensures it for each individual producer $j \in J$, and (3) ensures it for the economy as a whole.

Note especially that allocations with supplies exceeding demands, and so with surpluses that need to be disposed of, are not assumed to be automatically feasible. This is something of a departure from standard general equilibrium theory, which has customarily weakened the resource balance constraint (3) above to:

(3′) $\sum_{i \in I} x^i \leqq \sum_{j \in J} y^j$.

There are two reasons for preferring to work with (3) rather than with (3′), however. The first is some added realism, especially when we come to discuss externalities and public

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goods later on. It simply is not reasonable to assume that all surplus supplies can be dumped costlessly. A second reason is that no generality is lost anyway. For, if free disposal really is possible, we can accommodate it within the framework presented here by including within the set J an additional fictitious "disposal firm" d whose production set is assumed to be $Y^d := \{ y^d \in \Re^G \mid y^d \leqq 0 \}$.

To summarize, then: a feasible allocation is a pair $(\mathbf{x}^I, \mathbf{y}^J) \in \mathbf{X}^I \times \mathbf{Y}^J$ satisfying the *resource balance constraint* that $\sum_{i \in I} x^i = \sum_{j \in J} y^j$.

4.2. Pareto efficiency

We now want to define an efficient allocation. When looking at the whole economy, efficiency means that an allocation is not dominated by any other allocation; in other words, we shall compare different allocations. It is natural to base such comparisons on consumers' welfare; what producers can achieve is only a means to this end. And, of course, Paretian welfare economics under consumer sovereignty involves looking at consumers' preferences, and only these preferences.

Accordingly, a feasible allocation will be defined as (Pareto) efficient if there is no other feasible allocation which is Pareto superior. Formally, the feasible allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ is *(Pareto) efficient* if there is no alternative feasible allocation $(\mathbf{x}^I, \mathbf{y}^J)$ such that $x^i \; R^i \; \hat{x}^i$ for all $i \in I$, with $x^h \; P^h \; \hat{x}^h$ for some $h \in I$.

A feasible allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ is *weakly Pareto efficient* if there is no alternative feasible allocation $(\mathbf{x}^I, \mathbf{y}^J)$ such that $x^i \; P^i \; \hat{x}^i$ for all $i \in I$.

Thus, in order to be weakly Pareto efficient, a feasible allocation must simply have the property that there is no alternative which makes every consumer better off. To see the difference from Pareto efficiency, notice that a feasible allocation could be weakly but not strongly Pareto efficient if there were an alternative that made one or more consumers better off and no consumers worse off, but with no alternative that makes all consumers better off simultaneously. In particular, if one or more consumers are (globally) satiated in the distribution $\hat{\mathbf{x}}^I$, then the feasible allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ is automatically weakly Pareto efficient.

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5. Market Equilibrium

5.1. *Walrasian equilibrium*

An obvious starting point for discussing competitive market allocations is the Walrasian equilibrium model of pure exchange. In that model, for any given price vector p, each consumer $i \in I$ is allowed a consumption vector c^i whose value $p\,c^i$ at prices p does not exceed the value $p\,\omega^i$ of the initial endowment ω^i. Thus the budget constraint is $p\,c^i \leq p\,\omega^i$. Since the net trade vector x^i satisfies $x^i = c^i - \omega^i$, this budget constraint can be written more simply as $p\,x^i \leq 0$. In this case, then, each consumer i's *budget set* takes the form $B^i(p,0) = \{\,x^i \in X^i \mid p\,x^i \leq 0\,\}$.

In this economy of pure exchange, it is usual to allow free disposal because there is no aggregate production set in which disposal activities can be included. For the same reason, only semi-positive price vectors are allowed. Then a *Walrasian equilibrium* is an allocation (or distribution) $\hat{\mathbf{x}}^I$ and a price vector $p > 0$ such that:

(1) for every $i \in I$, one has $\hat{x}^i \in B^i(p,0)$ and $\hat{x}^i \; R^i \; x^i$ for all $x^i \in B^i(p,0)$;

(2) $\sum_i \hat{x}^i \leqq 0$.

Two successive extensions of the Walrasian model of pure exchange are commonplace. Both involve private production. In the first, every firm has a production set with constant returns to scale. This implies that no firm earns a profit in equilibrium, and so there are no profits to distribute. Accordingly each consumer i can still be faced with a budget constraint of the form $p\,x^i \leq 0$. A Walrasian equilibrium in such an economy consists of an allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J) \in \mathbf{X}^I \times \mathbf{Y}^J$ with $\sum_i \hat{x}^i = \sum_j \hat{y}^j$ and a price vector $p \neq 0$ such that (1) above is satisfied, and also:

(2′) for every $j \in J$ and every $y^j \in Y^j$, one has $p\,y^j \leq p\,\hat{y}^j$.

Note especially how the assumption of free disposal has now been abandoned once again.

In the second Walrasian model with private production, firms do not necessarily produce under constant returns to scale and so they may be making profits in equilibrium. These profits have to be distributed. It is usually assumed that there is a *private ownership economy*, in which each consumer $i \in I$ receives a fixed share θ^{ij} of the profits earned by each firm $j \in J$. Thus, given the price vector p, each consumer i faces a budget constraint

of the form

$$p\,x^i \leq w^i := \sum\nolimits_{j \in J} \theta^{ij}\,\pi^j$$

where w^i denotes i's "wealth" and, for each $j \in J$, firm j's profits are denoted by π^j. Of course, in order to ensure that all profits really are distributed — that $\sum_i w^i = \sum_j \pi^j$, in other words — it is necessary to have $\sum_i \theta^{ij} = 1$ for each $j \in J$. Some θ^{ij} could be negative, however, and many are likely to be zero.

Notice here that really each firm j makes a profit π^j which is a function of the price vector p. In fact there is a profit function $\pi^j(p) := \max\{\,p\,y^j \mid y^j \in Y^j\,\}$ indicating the maximum profit that firm j can earn for any given price vector $p \neq 0$. For some price vectors p, it is possible that $\pi^j(p)$ could be $+\infty$, or that the profit maximum could be unattainable. But such price vectors can never occur in Walrasian equilibrium anyway.

Now, since each firm j's maximum profit is a function $\pi^j(p)$ of the price vector p, so then is each consumer i's net wealth in the private ownership economy. In fact, given the shareholdings θ^{ij} $(i \in I,\ j \in J)$, each consumer $i \in I$ must always have a "net wealth function" $w^i(p)$ which, for all $p \neq 0$, is given by

$$w^i(p) \equiv \sum\nolimits_j \theta^{ij}\,\pi^j(p).$$

Although in a private ownership economy the net wealth functions $\mathbf{w}^I(p) := \langle w^i(p)\rangle_{i \in I}$ are derived from the shareholdings θ^{ij} $(i \in I,\ j \in J)$, there is no need to limit their scope to such economies. Nor, indeed, need only private ownership wealth functions be considered even if the economy is one with private production. It is quite possible, at least in principle, for governments or other authorities (such as charities) to mediate in the distribution of wealth and so to bring about rather more general functions $\mathbf{w}^I(p)$ that describe the distribution of net wealth between different consumers.

These functions, moreover, can also be used to describe the "lump-sum transfers" that figure so prominently in the classical literature of welfare economics. If $w^i(p) > 0$ then consumer i is effectively receiving a transfer, although it may be made up wholly or in part of profit (or dividend) wealth transfers from firms which are partly owned by i. If $w^i(p) < 0$ then i is paying a lump-sum tax. Indeed, even if $w^i(p) > 0$ but $w^i(p) < \sum_{j \in J} \theta^{ij}\,\pi^j(p)$ in a private ownership economy, then i is still paying a lump-sum tax of amount $\sum_{j \in J} \theta^{ij}\,\pi^j(p) - w^i(p)$. On the other hand, if $w^i(p) > \sum_{j \in J} \theta^{ij}\,\pi^j(p)$ then i

receives a lump-sum subsidy or transfer of amount $w^i(p) - \sum_{j \in J} \theta^{ij} \pi^j(p)$. A transfer is allowed to be negative, of course. Indeed, in an exchange economy, since overall budget balance requires $\sum_{i \in I} w^i(p) = 0$, the system of lump-sum transfers is trivial unless at least one consumer receives a negative transfer. The term "lump-sum transfer" is meant to cover all these cases, and to include any dividend, profit or other "unearned" wealth transfers as well. Recall that wealth earned from supplying labour is included as negative net expenditure in the expression $p\,x^i$.

Accordingly, a *(lump-sum) transfer system* $\mathbf{w}^I(p)$ is a profile of net wealth functions $w^i(p)$, one for each consumer $i \in I$, which are defined for all $p \neq 0$ and satisfy the following properties:

(1) $\sum_{i \in I} w^i(p) = \sum_{j \in J} \pi^j(p)$ (all $p > 0$), where $\pi^j(p)$ (each $j \in J$) denotes firm j's profit function;

(2) for every positive scalar λ, every price vector $p \neq 0$, and every consumer $i \in I$, one has
$$w^i(\lambda\,p) = \lambda\,w^i(p).$$

The first property is an overall budget constraint. It states that the aggregate net wealth of all consumers is equal to the aggregate profit of all producers, as must be true in any closed economy with only private production. The second property, which is customary in general equilibrium models, represents the "absence of *numéraire* illusion". If all prices double, then so should everybody's net wealth (be it positive or negative) — i.e., the transfer system should be homogenous of degree one. Both properties are satisfed, of course, in the usual Walrasian economies of pure exchange or of private production and private ownership.

So far, we have shown that some familiar wealth distribution mechanisms are particular lump-sum transfer systems, and also shown how lump-sum transfers can indeed be incorporated in such systems. It is worth making a few further observations.

First, notice that the lump-sum transfers are completely independent of consumers' market transactions. As such, they represent non-distortionary taxes and transfers, in the sense that marginal rates of substitution and marginal rates of product transformation will still be equated to price ratios even after such taxes and transfers have been introduced. It is true that lump-sum transfers are allowed to depend upon prices but, insofar as in a Walrasian economy no single agent has the power to determine prices, this price dependence is also non-distortionary.

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Second, notice that this dependence of lump-sum transfers on prices is actually an important and essential feature of any reasonable transfer mechanism. Insofar as profits feature in the transfer mechanisms, transfers must depend on prices anyway. Even if there is a unique Walrasian equilibrium allocation in a private ownership economy, the price system is determined only up to an arbitrary scalar factor. If all prices are doubled, equilibrium is preserved but only by doubling each consumer's wealth from profits. Even without profits, however, it still makes sense to have price-dependent transfers. For example, suppose that wealth is being transferred to help meet the essential needs of some deserving poor people, and that consumer prices increase suddenly with the result that their cost of living goes up substantially. Then a good transfer system should presumably respond to this by increasing the transfers to the poor in nominal terms in order to offer some protection against a decline in their real living standards. All index-linked schemes of welfare payments are presumably intended to do just that.

5.2. Compensated and uncompensated equilibrium

For each agent $i \in I$, each fixed wealth level w^i, and each price vector $p \neq 0$, define the *budget set*

$$B^i(p, w^i) := \{ x \in X^i \mid p\,x \leq w^i \}$$

of feasible net trade vectors satisfying the budget constraint. Note that, if no trade is feasible for consumer i (even though i may not be able to survive without trade), then $0 \in X^i$. In this case $B^i(p, w^i)$ is never empty when $w^i \geq 0$.

Next define, for every $i \in I$ and $p \neq 0$, the following three demand sets:

(i) the *uncompensated demand set*, given by

$$\xi^{Ui}(p, w^i) := \{ x \in B^i(p, w^i) \mid x' \in P^i(x) \Longrightarrow p\,x' > w^i \}$$
$$= \arg\max_x \{ R^i \mid x \in B^i(p, w^i) \};$$

(ii) the *compensated demand set*, given by

$$\xi^{Ci}(p, w^i) := \{ x \in B^i(p, w^i) \mid x' \in R^i(x) \Longrightarrow p\,x' \geq w^i \};$$

(iii) the *weak compensated demand set*, given by

$$\xi^{Wi}(p, w^i) := \{ x \in B^i(p, w^i) \mid x' \in P^i(x) \Longrightarrow p\,x' \geq w^i \}.$$

The term "compensated" reflects the idea that the consumer's utility, or real income, is being held fixed, and that compensation for any price changes is being achieved as cheaply as possible. Evidently the definitions just given imply that

$$\xi^{Ui}(p, w^i) \cup \xi^{Ci}(p, w^i) \subset \xi^{Wi}(p, w^i).$$

Establishing when $\xi^{Ci}(p, w^i) = \xi^{Ui}(p, w^i)$ turns out to be very important later on. The following lemma shows that, because of local non-satiation, demands of all three kinds always exhaust the budget, and also there is in fact never any need to consider weak compensated demands, since they become equal to compensated demands. Furthermore, uncompensated demands become compensated demands, though the converse is not true without additional assumptions.

LEMMA 5.1. *Whenever preferences are locally non-satiated, then it must be true that:*

(i) $x \in \xi^{Wi}(p, w^i) \implies p\, x = w^i$;

(ii) $\xi^{Wi}(p, w^i) = \xi^{Ci}(p, w^i)$;

(iii) $\xi^{Ui}(p, w^i) \subset \xi^{Ci}(p, w^i)$.

PROOF: (i) Suppose that x is any member of X^i satisfying $p\, x < w^i$. Now local non-satiation implies that x belongs to the closure cl $P^i(x)$ of $P^i(x)$. So there must also exist $x' \in P^i(x)$ close enough to x to ensure that $p\, x' < w^i$. Therefore $x \notin \xi^{Wi}(p, w^i)$. Conversely, $x \in \xi^{Wi}(p, w^i)$ must imply that $p\, x \geq w^i$. But since $x \in \xi^{Wi}(p, w^i)$ implies $x \in B^i(p, w^i)$ and so $p\, x \leq w^i$, it must actually be true that $x \in \xi^{Wi}(p, w^i)$ implies $p\, x = w^i$.

(ii) Suppose that $\hat{x} \in \xi^{Wi}(p, w^i)$. Take any $x' \in R^i(\hat{x})$. Then $P^i(x') \subset P^i(\hat{x})$ because preferences are transitive. Yet, as discused in Section 2.3, local non-satiation implies that $x' \in$ cl $P^i(x')$ and so that $x' \in$ cl $P^i(\hat{x})$. But by definition, $\hat{x} \in \xi^{Wi}(p, w^i)$ implies $p\, x \geq w^i$ for all $x \in P^i(\hat{x})$. In fact the same must also be true for all $x \in$ cl $P^i(\hat{x})$, including x'. Therefore we have proved that $x' \in R^i(\hat{x})$ implies $p\, x' \geq w^i$. This shows that $\hat{x} \in \xi^{Ci}(p, w^i)$. Because $\xi^{Ci}(p, w^i) \subset \xi^{Wi}(p, w^i)$ trivially, it follows that $\xi^{Wi}(p, w^i) = \xi^{Ci}(p, w^i)$.

(iii) Because $\xi^{Ui}(p, w^i) \subset \xi^{Wi}(p, w^i)$ trivially, the already proved result of part (ii) implies that $\xi^{Ui}(p, w^i) \subset \xi^{Ci}(p, w^i)$. ∎

An *uncompensated* (resp. *compensated*) *equilibrium relative to a transfer system* $\mathbf{w}^I(p)$ is a feasible allocation $(\mathbf{x}^I, \mathbf{y}^J)$ together with a price vector p such that, for all $i \in I$, both $p\, x^i = w^i(p)$ and $x^i \in \xi^{Ui}(p, w^i(p))$ (resp. $\xi^{Ci}(p, w^i(p))$).

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5.3. Competitive and compensated competitive allocations

In much of what follows, the precise way in which the wealth distribution is determined will turn out not to be important. Instead it will be enough to consider the unearned wealth of each consumer in equilibrium. The relevant concept of equilibrium is then having an allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J) \in \mathbf{X}^I \times \mathbf{Y}^J$ satisfying $\sum_{i \in I} \hat{x}^i = \sum_{j \in J} \hat{y}^j$ be *competitive* at a price vector $p \neq 0$ in the following sense:

(i) the distribution $\hat{\mathbf{x}}$ is *competitive* — i.e., for every $i \in I$, it must be true that $x^i \in P^i(\hat{x}^i)$ implies $p \, x^i > p \, \hat{x}^i$ (so that $\hat{x}^i$ is *competitive* for every consumer $i \in I$);

(ii) the production plan $\hat{\mathbf{y}}$ is *competitive* — i.e., for every $j \in J$, it must be true that $y^j \in Y^j$ implies $p \, y^j \leq p \, \hat{y}^j$ (so that $\hat{y}^j$ is *competitive* for every producer $j \in J$).

The corresponding relevant concept of compensated equilibrium is that an allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J) \in \mathbf{X}^I \times \mathbf{Y}^J$ satisfying $\sum_{i \in I} \hat{x}^i = \sum_{j \in J} \hat{y}^j$ be *compensated competitive* at $p \neq 0$ in the sense that (ii) above is satisfied, but (i) is replaced by:

(i′) the distribution $\hat{\mathbf{x}}$ is *compensated competitive* — i.e., for every $i \in I$, it must be true that $x^i \in R^i(\hat{x}^i)$ implies $p \, x^i \geq p \, \hat{x}^i$ (so that $\hat{x}^i$ is *compensated competitive* for every consumer $i \in I$).

An uncompensated (resp. compensated) Walrasian equilibrium relative to a transfer system $\mathbf{w}^I(\cdot)$ therefore consists of an allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ and a price vector $p \neq 0$ such that the allocation is competitive (resp. compensated competitive) at the price vector p and also, for every $i \in I$, the budget constraint $p \, \hat{x}^i = w^i(p)$ is satisfied.

The difference between compensated and uncompensated equilibrium is illustrated by the following two examples. The first is known as *Arrow's exceptional case* (see Arrow, 1951). The consumer's feasible set is taken to be the non-negative quadrant $X = \{ (x_1, x_2) \mid x_1, x_2 \geq 0 \}$. The indifference curves are assumed to be given by the equation $x_2 = (u - x_1)^2$ for $0 \leq x_1 \leq u$, where the parameter u can be taken as the relevant measure of utility. So all the indifference curves are parts of parabolae, as indicated in Fig. 2.

This consumer has strictly monotone, continuous, and convex preferences, as is easily checked. Yet trouble arises at net demand vectors of the form $(x_1, 0)$ with x_1 positive, such as the point A in the diagram. This net demand vector is clearly compensated competitive at any price vector of the form $(0, p_2)$ where $p_2 > 0$. To make A competitive at any price
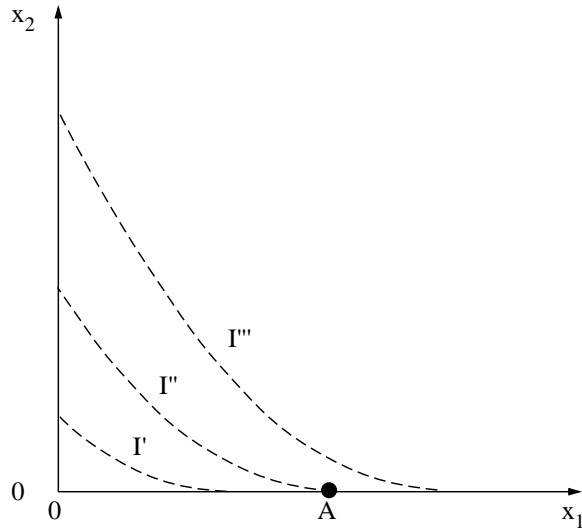
Figure 2

vector is impossible, however. For the price vector would have to take the form $(0, p_2)$ still, and so the budget constraint would have to be $p_2 x_2 \leq 0$ or $x_2 \leq 0$. But then the consumer could always move to preferred points by increasing x_1 while keeping $x_2 = 0$.

Another example of an allocation which is compensated competitive but not (uncompensated) competitive arises when the feasible set $X = \Re^2_+$ and preferences are "lexicographic" in the sense that

$$(x_1, x_2) \, R \, (x_1', x_2') \iff [x_1 > x_1'] \quad \text{or} \quad [x_1 = x_1' \quad \text{and} \quad x_2 \geq x_2'].$$

Consider any $\hat{x} \in X$ whose components $(\hat{x}_1, \hat{x}_2)$ are both positive. Then $\hat{x}$ must be compensated competitive at the price vector $p = (1, 0)$ because, if $x \, R \, \hat{x}$ then $x_1 \geq \hat{x}_1$ and so $p \, x \geq p \, \hat{x}$. But the preference ordering R obviously has no maximum on the budget line $p \, x = p \, \hat{x}$, which is $x_1 = \hat{x}_1$; by increasing x_2 indefinitely along this vertical budget line, the consumer moves to more and more preferred points.

The difficulty presented by lexicographic preferences is fairly easily excluded by assuming that preferences are continuous. In fact, it is enough to assume that every lower contour set $R^{i-}(x^i)$ is closed. Arrow's exceptional case, on the other hand, can be ruled out by assuming that each consumer $i \in I$ has a net trade vector $\hat{x}^i$ in the interior of the feasible set X^i. In this case we say that $\hat{\mathbf{x}}^I$ is an *interior distribution*. In order to prove

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that interiority is enough to ensure that a compensated competitive allocation is actually competitive, we begin with a more general result that will be used later in Section 7.

LEMMA 5.2 (THE CHEAPER POINT THEOREM). *Suppose that $\hat{x}^h$ is compensated competitive for consumer h at prices $p \neq 0$, but that $\underline{x}^h$ is a "cheaper point" of X^h with $p\,\underline{x}^h < p\,\hat{x}^h$. Suppose too that X^h is convex and that the lower contour set $R^{h-}(\hat{x}^h)$ is closed. Then $\hat{x}^h$ is competitive for consumer h.*
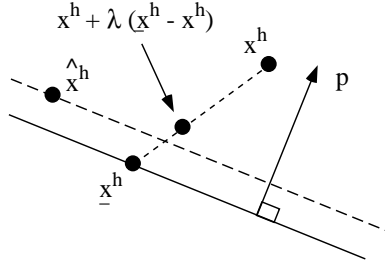

Figure 3

PROOF: Suppose that $x^h \in P^h(\hat{x}^h)$. Because X^h is convex and $R^{h-}(\hat{x}^h)$ is closed, there must exist λ with $0 < \lambda < 1$ such that

$$x^h + \lambda\,(\underline{x}^h - x^h) \in P^h(\hat{x}^h) \subset R^h(\hat{x}^h).$$

This is illustrated in Fig. 3. But then, by the hypothesis that $\hat{x}^h$ is compensated competitive, it follows that $p\,[x^h + \lambda\,(\underline{x}^h - x^h)] \geq p\,\hat{x}^h$, or equivalently that

$$(1 - \lambda)\,p\,x^h \geq p\,\hat{x}^h - \lambda p\,\underline{x}^h > (1 - \lambda)\,p\,\hat{x}^h.$$

The last strict inequality follows because $\lambda > 0$ and $p\,\underline{x}^h < p\,\hat{x}^h$. But then, dividing by $1 - \lambda$ which is positive, we obtain $p\,x^h > p\,\hat{x}^h$. ∎

PROPOSITION 5.3. *Suppose that each consumer has a convex feasible set and continuous preferences. Then, if $\hat{\mathbf{x}}^I$ is any interior distribution which is compensated competitive at prices $p \neq 0$, it must be competitive at prices p.*

PROOF: Suppose that some consumer $i \in I$ has a net demand vector $\hat{x}^i$ that is not competitive at prices p. Then there exists $\tilde{x}^i \in P^i(x^i)$ such that $p\,\tilde{x}^i \leq p\,\hat{x}^i$.[4] Because $\hat{x}^i \in$ int X^i and $p \neq 0$, there certainly exists a cheaper point $\underline{x}^i \in X^i$ such that $p\,\underline{x}^i < p\,\hat{x}^i$. So Lemma 5.2 applies. ∎

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[4] In fact $p\,\tilde{x}^i < p\,\hat{x}^i$ is impossible because $\hat{x}^i$ is compensated competitive. Therefore $p\,\tilde{x}^i = p\,\hat{x}^i$. Yet only $p\,\tilde{x}^i \leq p\,\hat{x}^i$ is needed for the proof which follows.

6. First Efficiency Theorem

6.1. Weak efficiency

In this section it will be shown first that a competitive allocation is weakly Pareto efficient and, if all consumers have locally non-satiated preferences, (fully) Pareto efficient.

LEMMA 6.1. *Suppose that the allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ is competitive at prices $p \neq 0$. Then there is no feasible allocation $(\mathbf{x}^I, \mathbf{y}^J)$ such that $p \sum_i x^i > p \sum_i \hat{x}^i$.*

PROOF: By hypothesis, the production plan $\hat{\mathbf{y}}^J$ is competitive at prices p. Therefore, if $\mathbf{y}^J \in \mathbf{Y}^J$, then $p\,y^j \leq p\,\hat{y}^j$ for all $j \in J$, which implies that $p \sum_j y^j \leq p \sum_j \hat{y}^j$. So, if $(\mathbf{x}^I, \mathbf{y}^J) \in \mathbf{X}^I \times \mathbf{Y}^J$ is any feasible allocation with $\sum_i x^i = \sum_j y^j$, then

$$ p \sum_i x^i = p \sum_j y^j \leq p \sum_j \hat{y}^j = p \sum_i \hat{x}^i $$

and so $p \sum_i x^i \leq p \sum_i \hat{x}^i$. ∎

Without assuming local non-satiation or anything else, this gives:[5]

PROPOSITION 6.2. *Any competitive allocation is weakly Pareto efficient.*

PROOF: Suppose that $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ is an allocation which is competitive at prices $p \neq 0$. If the distribution $\mathbf{x}^I$ is strictly Pareto superior, then $x^i \in P^i(\hat{x}^i)$ for all $i \in I$, and so $p\,x^i > p\,\hat{x}^i$. This implies that $p \sum_i x^i > p \sum_i \hat{x}^i$. By Lemma 6.1, it follows that there can be no feasible allocation $(\mathbf{x}^I, \mathbf{y}^J)$ with distribution $\mathbf{x}^I$. So no feasible allocation $\mathbf{x}^I$ can be strictly Pareto superior after all. ∎

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[5] Here, the two assumptions that the set of individuals and the set of goods are both finite play an important role. Otherwise, if both assumptions are relaxed together, as they are in overlapping generations economies, a competitive allocation need not be even weakly Pareto efficient. For more discussion of the overlapping generations model originally due to Allais (1947) and Samuelson (1958), see the surveys by Geanakoplos (1987) and by Geanakoplos and Polemarchakis (1991).
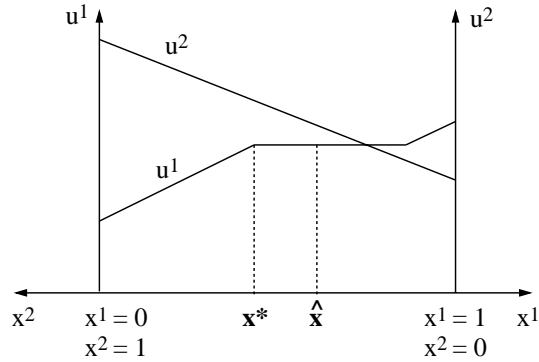
Figure 4

6.2. *Failure of efficiency*

Nevertheless, it is not generally true that any competitive allocation is efficient, rather than merely weakly efficient. This can be seen from a very simple example of an exchange economy involving just two consumers with weakly monotone preferences and a single good, as illustrated in Fig. 4.

A feasible allocation is represented by a point such as $\mathbf{x}^*$ or $\hat{\mathbf{x}}$ on the line segment joining the two extreme allocations $(0, 1)$ and $(1, 0)$ — the usual Edgeworth box has collapsed to a line interval. The axes labelled u^1 and u^2 represent particular ordinal measures of utility for the two individuals. Any non-wasteful allocation $\hat{\mathbf{x}} = (\hat{x}^1, \hat{x}^2)$ with $\hat{x}^1 + \hat{x}^2 = 1$ is competitive at the price 1 (for the one good) and wealth distribution $\mathbf{w} = (x^1, x^2)$. Suppose that, as indicated in the diagram, Consumer 1 is locally satiated at $\mathbf{x}^*$ whereas Consumer 2 is never satiated. Then the competitive allocation $\hat{\mathbf{x}}^I$ is inefficient because moving from $\hat{\mathbf{x}}^I$ to $\mathbf{x}^*$ makes Consumer 2 better off, while leaving Consumer 1 indifferent. The trouble is that taking away small amounts of the one consumption good makes Consumer 1 no worse off.

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6.3. Local non-satiation

This problem can be overcome with the extra assumption that all consumers have locally non-satiated preferences. Indeed, the local non-satiation assumption implies a useful extra property of any competitive allocation:

LEMMA 6.3. *Suppose that the consumer i has a feasible set X^i and preference ordering R^i satisfying local non-satiation. Then, if $\hat{x}^i$ is competitive for consumer i at prices $p \neq 0$, it is also compensated competitive.*

PROOF: Suppose $\hat{x}^i$ is not compensated competitive for consumer i at prices $p \neq 0$. Then there exists $\bar{x}^i \in R^i(\hat{x}^i)$ such that $p\,\bar{x}^i < p\,\hat{x}^i$. But then there must also be a neighbourhood N of $\bar{x}^i$ such that $p\,x^i \leq p\,\hat{x}^i$ for all $x^i \in N$. Because of local non-satiation at $\bar{x}^i$, there exists $\tilde{x}^i \in N$ such that $\tilde{x}^i \in P^i(\bar{x}^i)$. This implies that $\tilde{x}^i \in P^i(\hat{x}^i)$ because $\tilde{x}^i\ P^i\ \bar{x}^i\ R^i\ \hat{x}^i$ and R^i is transitive. Yet $p\,\tilde{x}^i \leq p\,\hat{x}^i$ because $\tilde{x}^i \in N$. So $\hat{x}^i$ cannot be competitive for consumer i.

Conversely, if $\hat{x}^i$ is competitive for i, then it must be compensated competitive. ∎

We also have:

LEMMA 6.4. *If the feasible allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ is both competitive and compensated competitive at the same price vector $p \neq 0$ then:*

(a) $\sum_i p\,x^i > \sum_i p\,\hat{x}^i$ *for any Pareto superior distribution $\mathbf{x}$;*

(b) $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ *is efficient.*

PROOF: Let $\mathbf{x}^I$ be any distribution that is Pareto superior to $\hat{\mathbf{x}}^I$. Then:

(a) By definition, $x^i\ R^i\ \hat{x}^i$ for all $i \in I$ and $x^h\ P^h\ \hat{x}^h$ for some $h \in I$. Because $\hat{\mathbf{x}}^I$ is competitive, it follows that $p\,x^h > p\,\hat{x}^h$. Also, because $\hat{\mathbf{x}}^I$ is compensated competitive, it must be true that $p\,x^i \geq p\,\hat{x}^i$ for all $i \in I$. So adding over all consumers gives $\sum_i p\,x^i > \sum_i p\,\hat{x}^i$.

(b) By Lemma 6.1, the conclusion of (a) evidently implies that there is no feasible allocation of the form $(\mathbf{x}^I, \mathbf{y}^J)$. Hence no feasible allocation can be Pareto superior to $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$, which must therefore be efficient. ∎

Combining Lemmas 6.3 and 6.4 (b) gives:

PROPOSITION 6.5. *If all consumers' preferences are locally non-satiated, then any competitive allocation is efficient.*

7. When Efficient Allocations are Compensated Competitive

Section 6 showed that any competitive allocation is weakly Pareto efficient, and also that local non-satiation of preferences is sufficient to ensure that any competitive allocation is Pareto efficient. For the converse to be true, however, and for any Pareto efficient distribution to be competitive, stronger assumptions are generally required. To begin with, as can be seen from a simple Edgeworth box diagram for an exchange economy with two goods and two consumers, it is unlikely that a *particular* Pareto efficient allocation on the "contract curve" can be sustained as a Walrasian equilibrium, even though it may be competitive. As discussed in the introduction, the reason is that the distribution of wealth is unlikely to be appropriate.

So this section will be concerned with showing that every Pareto efficient allocation is competitive, but only for a suitable distribution of income. In order that even this can be true, however, a number of additional assumptions will have to be made. Indeed, in the case of a single consumer, one first needs convexity in production, as is shown by the example illustrated in Fig. 5.
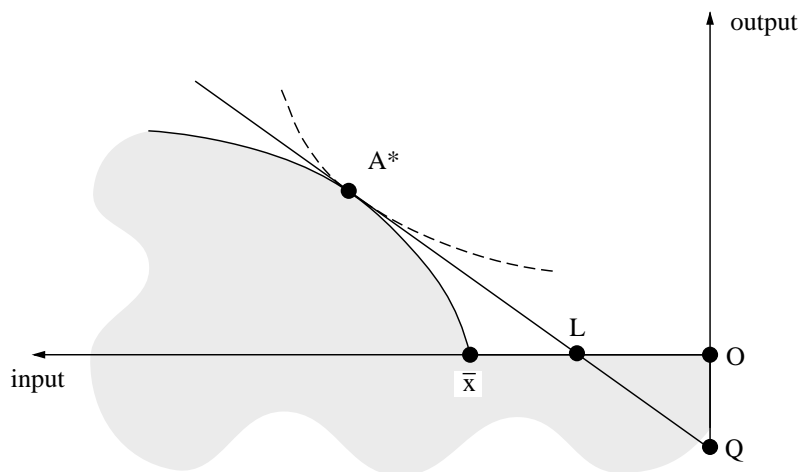

Figure 5

Here a single producer uses just one input to produce a single output. The producer is assumed to have a production set with free disposal, as indicated by the shaded region. Unless at least the quantity $\bar{x}$ of the single input is used, output must be zero. So there are fixed costs. Moreover, the point A^* is on the production frontier, and is efficient. It

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may even be optimal in the sense that, among all feasible allocations, it maximizes the preference ordering of the only consumer. This is even suggested by the indifference curve which has been included in the diagram. Yet, if one sets prices corresponding to the slope of the tangent to the production frontier at A^*, the producer maximizes profit by choosing the origin O rather than the point A^*. Indeed, at these prices, the producer at A^* faces a loss whose extent is equivalent to giving up either OL units of input or OQ units of output.

Such difficulties are usually avoided by assuming that the aggregate production set Y is convex. Along with convexity of the aggregate production set, however, there is also a need for convexity in consumers' feasible sets and in their preferences. Otherwise there could be difficulties similar to those illustrated in Fig. 5 even in an Edgeworth box exchange economy. In Section 8 other assumptions will also be required in order to ensure that a Pareto efficient allocation is competitive. For the moment, we begin by showing that such allocations are at least compensated competitive.

PROPOSITION 7.1. *If all consumers have locally non-satiated convex preferences, and if the aggregate production set is convex, then any weakly Pareto efficient allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ is compensated competitive at some price vector $p \neq 0$.*

PROOF: (1) Because the allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ is weakly Pareto efficient, the aggregate production set $Y = \sum_j Y^j$ and the aggregate preference set $\sum_i P^i(\hat{x}^i)$ must be disjoint. For otherwise there would exist a feasible allocation $(\mathbf{x}^I, \mathbf{y}^J) \in \mathbf{X}^I \times \mathbf{Y}^J$ with $\sum_i x^i = \sum_j y^j \in Y$ and $x^i \in P^i(\hat{x}^i)$ for all $i \in I$, in which case $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ could not be even weakly Pareto efficient.

(2) By Prop. 2.1, convex preferences imply that $P^i(\hat{x}^i)$ is convex for each i. But the sum of convex sets is always convex.[6] So the two sets Y and $\sum_i P^i(\hat{x}^i)$ are disjoint non-empty convex sets. They can therefore be separated by a hyperplane $p\,z = \alpha$ (with $p \neq 0$)

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[6] This is well known, but here is a proof anyway. Suppose that K^i $(i \in I)$ is a finite collection of convex sets. Suppose that $K = \sum_i K^i$ and that $c = \lambda a + \mu b$ is a convex combination of two points $a, b \in K$, where λ and μ are non-negative convex weights satisfying $\lambda + \mu = 1$. Then there exist $a^i, b^i \in K^i$ $(i \in I)$ such that $a = \sum_i a^i$ and $b = \sum_i b^i$. Now

$$c = \lambda a + \mu b = \lambda \sum_i a^i + \mu \sum_i b^i = \sum_i (\lambda a^i + \mu b^i) = \sum_i c^i$$

where $c^i = \lambda a^i + \mu b^i$ for all $i \in I$. But because each K^i is convex, it follows that $c^i \in K^i$ $(i \in I)$. Since $\sum_i c^i = c$, it must be true that $c \in K$.
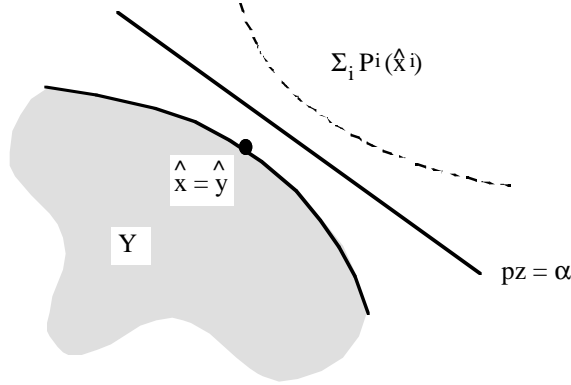
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$\Sigma_i P^i(\hat{x}^i)$

$\hat{x} = \hat{y}$

Y

$pz = \alpha$

Figure 6

in the commodity space $\Re^G$. Specifically, as shown in Fig. 6, there exist $p \neq 0$ and α such that $p\,y \leq \alpha$ for all $y \in Y$ and $p\,x \geq \alpha$ for all $x \in \sum_i P^i(\hat{x}^i)$.

(3) Let $R(\hat{\mathbf{x}}^I) := \sum_i R^i(\hat{x}^i)$. Suppose that $x \in R(\hat{\mathbf{x}}^I)$. Then there exists $\mathbf{x}^I \in \mathbf{X}^I$ such that $x = \sum_i x^i$ and $x^i\ R^i\ \hat{x}^i$ (all $i \in I$). Because every consumer's preferences are locally non-satiated, for every $\epsilon > 0$ and every consumer i there exists $x^i(\epsilon) \in P^i(x^i)$ near enough to x^i so that $p\,x^i(\epsilon) \leq p\,x^i + \epsilon/\#I$, where $\#I$ is the number of consumers. So, adding over all consumers, it follows that

$$p\,x = \sum_i p\,x^i \geq \sum_i \left[p\,x^i(\epsilon) - \frac{\epsilon}{\#I}\right] = p\,x(\epsilon) - \epsilon$$

where $x(\epsilon) := \sum_i x^i(\epsilon)$.

(4) But for each $i \in I$ one has $x^i(\epsilon)\ P^i\ x^i$ and $x^i\ R^i\ \hat{x}^i$. Since R^i is transitive, it follows that $x^i(\epsilon) \in P^i(\hat{x}^i)$. Therefore $x(\epsilon) \in \sum_i P^i(\hat{x}^i)$. From (2) it follows that $p\,x(\epsilon) \geq \alpha$. Then (3) implies that $p\,x \geq \alpha - \epsilon$. Since this must be true for every $\epsilon > 0$ and every $x \in R(\hat{\mathbf{x}}^I)$, it follows that $p\,x \geq \alpha$ for all such x.

(5) Let $\hat{x} := \sum_i \hat{x}^i$ and $\hat{y} := \sum_j \hat{y}^j$. Because preferences are reflexive, $\hat{x}^i \in R^i(\hat{x}^i)$ for all $i \in I$. Therefore $\hat{x} \in R(\hat{\mathbf{x}}^I)$ and $\hat{y} \in Y$ so that, by (2) and (4), $p\,\hat{x} \geq \alpha \geq p\,\hat{y}$. But $\hat{x} = \hat{y}$ because of feasibility, and so $p\,\hat{x} = p\,\hat{y} = \alpha$. That is, the hyperplane $p\,z = \alpha$ must actually pass through both $\hat{x}$ and $\hat{y}$, as shown in Fig. 7.

(6) It follows from (2) and (5) that $p\,(y - \hat{y}) = \sum_j p\,(y^j - \hat{y}^j) \leq 0$ for all $y \in Y$ and so for all $\mathbf{y}^J \in \mathbf{Y}^J$. Now, for each $k \in J$, any production plan $\mathbf{y}^J = (y^j)_{j \in J}$ with $y^k \in Y^k$ and $y^j = \hat{y}^j$ for all $j \in J \setminus \{k\}$ is certainly a member of $\mathbf{Y}^J$. For each $k \in J$, it follows that $y^k \in Y^k$ implies $p\,(y^k - \hat{y}^k) \leq 0$. This confirms that $\hat{\mathbf{y}}^J$ must be a competitive production plan.

(7) Also (4) and (5) above imply that $p\,x \geq p\,\hat{x}$ for all $x \in R(\hat{\mathbf{x}}^I)$. So $\sum_i p\,(x^i - \hat{x}^i) = p\,(x - \hat{x}) \geq 0$ for all $\mathbf{x}^I \in \prod_i R^i(\hat{x}^i)$. But for all $h \in I$, it is obviously true that $\hat{x}^i \in R^i(\hat{x}^i)$
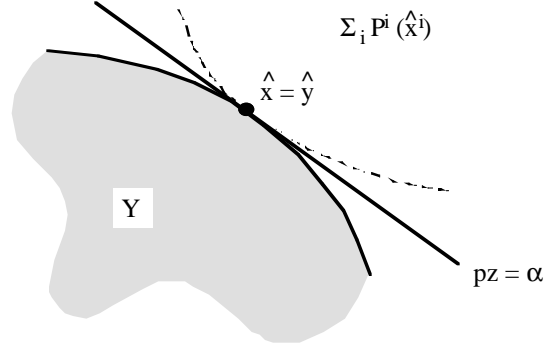
Figure 7

for all $i \in I \setminus \{h\}$, because preferences are reflexive. So, when $x^h \in R^h(\hat{x}^h)$ and $x^i = \hat{x}^i$ for all $i \in I \setminus \{h\}$, it must be true that $\mathbf{x}^I \in \prod_{i \in I} R^i(\hat{x}^i)$, and so

$$0 \leq \sum_{i \in I} p\,(x^i - \hat{x}^i) = p\,(x^h - \hat{x}^h) + \sum_{i \in I \setminus \{h\}} p\,(x^i - \hat{x}^i) = p\,(x^h - \hat{x}^h).$$

Therefore $x^h \in R^h(\hat{x}^h)$ implies $p\,x^h \geq p\,\hat{x}^h$, for every $h \in I$. This confirms that the distribution $\mathbf{x}^I$ must be compensated competitive.

(8) From (6) and (7) it follows that the allocation $(\mathbf{x}^I, \mathbf{y}^J)$ as a whole must be compensated competitive. ∎

8.   The Second Efficiency Theorem

8.1.   Relevant commodities

Arrow's exceptional case was presented in Section 5.3. So was the interiority assumption that $\hat{x}^i \in \text{int } X^i$ (all $i \in I$), which is often introduced to rule out this troublesome example. This interiority assumption is unacceptably strong, however, insofar as it requires every consumer to consume positive amounts of all those consumption goods which cannot be produced domestically and sold. Yet no efficient distribution can have this property in an economy where there is any consumer with no desire at all for some consumption good that another consumer wants. For efficiency then requires that a consumer with no desire for such a good should not be consuming it at all, nor demanding it.

Moreover, even Arrow's example seems somewhat contrived in that good 2 plays no real role in that economy. Indeed, it can never be traded because it is in zero supply and the lone consumer cannot consume a negative amount. I propose to exclude Arrow's exceptional

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case by regarding any goods which can never be traded as irrelevant and concentrating only on the space of relevant commodities. Specifically, let $V := \sum_j Y^j - \sum_i X^i$ denote the set of net export vectors which could be provided to the rest of the world if the economy somehow became open to trade from outside. Then it is assumed that 0 lies in the interior of the set V. This interiority condition implies in particular that, for each good g and the corresponding unit vector e_g with one unit of good g and nothing of any other good, there exists a small enough $\epsilon > 0$ such that both $\epsilon\, e_g$ and $-\epsilon\, e_g$ belong to V. Thus the economy is capable of absorbing a positive net import of each good, as well as of providing a positive net export of each good. In other words, there is enough slack in the economy to allow trade in each direction in all goods. In this case it is said that *all goods are relevant.*

PROPOSITION 8.1. *Suppose that the feasible allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ is compensated competitive at prices $p \neq 0$, and that $0 \in$ int V (where $V := \sum_j Y^j - \sum_i X^i$). Then there exists at least one consumer h for whom $\hat{x}^h$ is not a cheapest point of the feasible set X^h.*

PROOF: Suppose, on the contrary, that $\hat{x}^i$ is a cheapest point of X^i for every consumer $i \in I$, so that $p\, x^i \geq p\, \hat{x}^i$ for all $x^i \in X^i$. Now, whenever $v \in V$, there exist $x^i \in X^i$ (all i) and $y^j \in Y^j$ (all j) such that $v = \sum_j y^j - \sum_i x^i$. Then $p\, x^i \geq p\, \hat{x}^i$ (all i) and $p\, y^j \leq p\, \hat{y}^j$ (all j) because $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ is compensated competitive at prices p. So

$$p\, v = \sum_j p\, y^j - \sum_i p\, x^i \leq \sum_j p\, \hat{y}^j - \sum_i p\, \hat{x}^i = p \left(\sum_j \hat{y}^j - \sum_i \hat{x}^i \right) = 0,$$

where the last equality holds because $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ is feasible. Therefore $p\, v \leq 0$ for all $v \in V$, where $p \neq 0$. This implies that 0 must be on the boundary of V.

Conversely, the assumption that $0 \in$ int V implies that at least one consumer must not be at a cheapest point. ∎

So far, then, it has been established that at least one consumer $h \in I$ is not at a cheapest point of the feasible set X^h. By Lemma 5.2, the net demand $\hat{x}^h$ of this consumer is competitive. Next, a condition will be found to guarantee that *every* consumer's net demand is competitive because no consumer is at a cheapest point.

8.2. Non-oligarchic allocations

Some of the force of Arrow's exceptional case, which was presented in Section 5.3, has already been blunted by assuming that all goods are relevant, meaning that $0 \in$ int $(\sum_j Y^j - \sum_i X^i)$. In particular, Lemma 5.2 and Prop. 8.1 together show that the Arrow exceptional case cannot occur in a one consumer economy in which all commodities are relevant. But with many consumers some difficulties may still remain, as shown by:
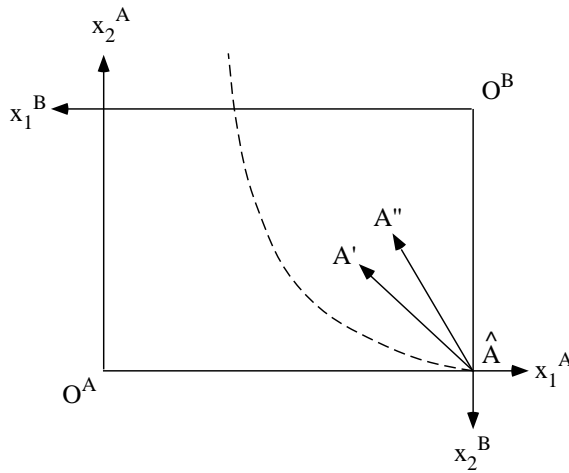

Figure 8

EXAMPLE. *Consider the pure exchange economy with two goods and two consumers, as illustrated by the Edgeworth box diagram of Fig. 8. Suppose that consumer B has horizontal indifference curves, while one of consumer A's indifference curves is as drawn, with a horizontal tangent at the point $\hat{A}$.*

The allocation $\hat{A}$ is Pareto efficient in this example. And the relevant commodity space is $\Re^2$ because the exchanges to A' and A'', for instance, are both feasible, and these two vectors span the whole of $\Re^2$. Obviously, the only price vectors at which $\hat{A}$ is compensated competitive take the form $(0, p_2)$ for $p_2 > 0$. But consumer A is in Arrow's exceptional case and so the allocation $\hat{A}$ is not competitive at any price vector.

The problem in this example is that, although allocation $\hat{A}$ does not occur at a cheapest point for consumer B, it does for consumer A. Moreover, A can only offer good 1 to consumer B, which in fact B does not care for.

With this example in mind, for any proper subset H of the set of consumers I, say that H is an *oligarchy* at the feasible allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ provided that there is no alternative feasible allocation $(\mathbf{x}^I, \mathbf{y}^J)$ satisfying $x^i \in P^i(\hat{x}^i)$ for all $i \in H$. Thus, when H is an oligarchy, it monopolizes resources to such an extent that no redistribution of resources from outside H could possibly bring about a new allocation making all the members of H better off simultaneously. Of course, in the example of Fig. 8, the allocation $\hat{A}$ at the corner of the Edgeworth box is certainly oligarchic. Indeed, consumer B is really a "dictator" at $\hat{A}$, inasmuch as no other feasible allocation in the box could make B better off.

On the other hand, say that the feasible allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ is *non-oligarchic* provided that, whenever H is a proper subset of I, then H is not an oligarchy at $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$. Then, no matter how the consumers are divided into two non-empty groups, each group is able to benefit strictly from resources which the other complementary group is able to provide. As discussed in Hammond (1993), this non-oligarchy assumption is related to McKenzie's (1981) concept of "irreducibility."

LEMMA 8.2. *If each consumer has a convex feasible set and continuous preferences, and if all commodities are relevant, then any non-oligarchic feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ which is compensated competitive at prices $p \neq 0$ must also be competitive at these prices.*

PROOF: (1) Because $0 \in \text{int}\,(\sum_j Y^j - \sum_i X^i)$, Prop. 8.1 implies that there exists at least one consumer $h \in I$ for whom $\hat{x}^h$ is not a cheapest point of X^h.

(2) Suppose that the non-empty set $H \subset I$ consists only of individuals i who do not have $\hat{x}^i$ as a cheapest point of X^i at prices p. Suppose too that $H \neq I$. Because H cannot be an oligarchy, there exists an alternative feasible allocation $(\mathbf{x}^I, \mathbf{y}^J)$ such that $x^i \in P^i(\hat{x}^i)$ for all $i \in H$. Then, because of the cheaper point Lemma 5.2, $p\,x^i > p\,\hat{x}^i$ for all $i \in H$, and so $\sum_{i \in H} p\,\hat{x}^i < \sum_{i \in H} p\,x^i$. Therefore

$$\sum\nolimits_{i \in H} p\,\hat{x}^i + \sum\nolimits_{i \in I \setminus H} p\,x^i < \sum\nolimits_{i \in I} p\,x^i = \sum\nolimits_{j \in J} p\,y^j \leq \sum\nolimits_{j \in J} p\,\hat{y}^j = \sum\nolimits_{i \in I} p\,\hat{x}^i$$

implying that $\sum_{i \in I \setminus H} p\,x^i < \sum_{i \in I \setminus H} p\,\hat{x}^i$. Thus, whenever $H \neq I$ is a set of individuals with cheaper points, there always exists at least one other individual $i \in I \setminus H$ with a cheaper point.

(3) Let H^* denote the set of all individuals i for whom $\hat{x}^i$ is not a cheapest point of X^i at prices p. From (1), it follows that $H^* \neq \emptyset$, and then from (2), that $H^* = I$.

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(4) For every $i \in I$, therefore, $\hat{x}^i$ is not a cheapest point of X^i. So $\hat{x}^i$ is competitive for consumer i, by Lemma 5.2. Thus, the allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ must be competitive at prices $p \neq 0$.　▌

8.3.　Second efficiency theorem: general version

PROPOSITION 8.3.　*Suppose that all commodities are relevant, and that $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ is a weakly Pareto efficient and non-oligarchic feasible allocation in an economy with consumers who all have convex, continuous and locally non-satiated preferences, while the aggregate production set is convex. Then there exists a price vector $p \neq 0$ at which $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ is competitive.*

PROOF: By Proposition 7.1, there exists a price vector $p \neq 0$ at which the feasible allocation $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ is compensated competitive. By Lemma 8.3, it follows at once that $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J)$ is competitive.　▌

In Section 4, both weakly Pareto efficient and Pareto efficient allocations were defined, and the contrast between them was illustrated. Yet now, under the hypotheses of Prop. 8.3 (which include local non-satiation), any regular weakly Pareto efficient allocation is competitive. On the other hand, local non-satiation implies that any competitive allocation is (fully) Pareto efficient, by Prop. 6.5. So, under those same hypotheses, it follows that any regular weakly Pareto efficient allocation is actually (fully) Pareto efficient.

9.　Externalities and the Public Environment

9.1.　Introduction

Sections 6 and 8 presented the two fundamental efficiency theorems relating Walrasian equilibrium or competitive allocations to allocations which are Pareto efficient. The first efficiency theorem in particular assures us that markets can usually be expected to achieve a Pareto efficient allocation, at least in the case when there is perfect competition within a complete market system. In the light of these results, it is therefore usual to regard any instance of Pareto inefficiency in a market economy as some kind of "market failure". Yet the results do depend upon the absence of a number of important obstacles which, in any actual economy, are likely to stand in the way of Pareto efficiency. These will be briefly discussed in the next three sections of this chapter, along with the remedies which may be available.

The first kind of obstacle to Pareto efficiency occurs when market equilibrium is Pareto inefficient, either because some goods have been left out of account and not traded at all, or because some producers or perhaps even some groups of consumers or workers behave in a way which is inconsistent with Walrasian equilibrium. Market failures of this kind seem to arise in connection with public goods (or the "public environment") and with external effects. These will be the topic of this Section.

9.2. Allocations with public environments

By public goods and services we mean those goods which are enjoyed not just by one consumer (individual or household) who has sole access to them, but which benefit a whole community of different consumers. These include "pure" public goods such as radio or television broadcasts (not, however, broadcasts on cable television) where, if any one household is sent the signal which enables it to receive the broadcast, then all the neighbours will also be able to receive the broadcast from the same signal. Streetlighting is a similar example.

There is also a much broader category of "impure" public goods and services. In principle these could be provided privately, with consumers paying for them and then also retaining them for their own private use. Yet in practice they get provided at no charge to anybody who cares to use them. Such goods include roads, footpaths, public parks, and, in many countries at least, basic education and health services. Other relatively pure public goods, where private provision would lead to serious constitutional and political difficulties in modern states, include police, justice, tax gathering, and the armed forces. Of course some public goods can be privatized, and *vice versa* — as, for example, in the rather surprising case of prison services, where there are experiments currently in Britain, France, and the U.S. allowing private firms to run some prisons under contracts with public authorities. In fact public goods are an important part of modern developed economies, since their provision certainly absorbs rather over 20% of gross domestic product in most countries.[7] So it is important to include public goods in our description of an economic allocation. This is partly for the negative reason that their provision absorbs a good deal

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[7] The much higher percentage which is usually quoted for *public expenditure* includes many items such as social security payments and debt servicing which are really transfer payments rather than public goods. Indeed, according to the *Economic Report of the President, February 1995*, in 1994 the U.S. annual gross domestic product was $6.737 trillion (see p. 274), of which total expenditure

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of the economy's total resources, but also for the positive reason that public goods are an essential feature of any modern economy with their own intrinsic value.

In order to describe the provision of public goods, it is necessary to expand the commodity space. Previously, we have been considering just allocations of private goods in the set G; now we shall add some extra public goods in a new set H. Typically government — both central and local or regional — will provide a vector of public goods $z \in \Re^H$. Producing these usually requires costly private goods. To describe these production activities, some extra "public producers" with their own net output vectors for private goods should be introduced. Typically their net output vectors will have negative components, because the public sector requires inputs of private goods in order to produce public goods. But there is no reason to exclude *a priori* the public production of private goods.

It is important to interpret the levels of public good provision z clearly. If one particular component g denotes "public health services," it is *not* the case that z_g is the amount of health care which everybody is forced to consume. Rather, z_g indicates the general level of *availability* and *quality* of health services which any individual has access to in case of need (cf. Drèze and Hagen, 1978). This interpretation should be borne in mind carefully, especially when considering economies with very large populations. Then it becomes natural to think of z_g as indicating the availability of health care to the typical person, and to think of the cost per head of providing z_g as being roughly constant for all sizes of population beyond a certain basic size. But much of the public environment does not consist of such public goods. It is convenient to include within the vector z many other facets of economic life. Thus, enjoyment of a public park will depend not only on the size of the park and the quality of its facilities, but also on the number of other people who are in the park at the same time. Insofar as that number is influenced or determined by economic decisions — including perhaps the decision of the park authorities to limit entry at peak times — it too is a component of z. Such a park is an example of a "public good with exclusion" in the sense of Drèze (1980).

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by Federal and State and local governments on purchases of goods and services was \$1.175 trillion (see p. 373), or 17.44% of G.D.P. But total expenditure by Federal and State and local governments on all items, including transfers and net interest and dividend payments, was \$2.257 trillion, or 33.50% of G.D.P.

In addition, there are cases when some purely private goods should probably be treated as components of z. Take for instance a firm that produces under increasing returns to scale, so that its production set Y^j is non-convex. Let one component of z represent the single output of this firm. Suppose that all the firm's isoquants are convex. Then, in this commodity space, the production set $Y^j(z)$ of net output vectors y^j which are feasible given z will be convex, and this may suffice to ensure that any Pareto efficient allocation will be competitive, given that the firm is unable to vary z. At least the production non-convexity is overcome, though at the cost of making the choice of the firm's output a public decision. In effect, however, it is a public decision anyway, as suggested in Section 11.2 below. This illustrates how there is always some kind of public good aspect to any economic decision which cannot easily be left to markets.

9.3. Externalities

"External effects", "externalities", and "spillover effects" are all terms commonly used by economists to describe what is essentially one and the same phenomenon. Intuitively, the idea is that consumers and firms undertake certain activities which affect those around them. But the carrying out of these activities is not subject to any discipline from the market. This is essentially what distinguishes externalities from ordinary economic activities such as hiring capital and labour at factor prices determined in competitive markets in order to produce an output which is then sold in another competitive market. In practice externalities arise in numerous different ways — see, for example, the discussion by Heller and Starrett (1976), as well as other articles in the same volume Lin (1976). A precise definition is surprisingly hard to formulate. This is probably because, if externalities really do matter, one is forced to ask why a suitable market does not get set up in order to allocate them efficiently, just as markets can allocate ordinary commodities. Nevertheless, I shall put forward here a general model which seems able to capture the most important features of the common examples of externalities, such as noise, pollution, congestion, absence of property rights, missing markets, etc. The last Section of this chapter considers at least one reason why markets for externalities may be difficult to arrange.

Although this has not been generally recognized, externalities and public goods are actually very closely related to each other. We saw in Section 9.2 how public goods, or the "public environment", are things which affect all consumers and all producers in the econ-

omy. The same is true, in principle, of externalities. Of course, many forms of atmospheric pollution in the Northern hemisphere have little impact on the Southern hemisphere. But it is still possible to regard all forms of pollution everywhere as parts of a world environment, even though most people will be affected much more by pollution in their own region than by pollution in distant regions. This is just the same as realizing that the quality of publicly provided local health services in the next city is of much less importance than that in the city where one lives.

In fact, it turns out that the main difference between externalities and public goods arises because of who provides or creates them. Moreover the current state of the world is such that it is easier to think of externalities as agents damaging each other by polluting the air, creating noise, etc. instead of helping each other as they do in the famous example of the apple-grower and the bee-keeper. There the bees feed on nectar from the apple trees while simultaneously pollenating those trees — in effect, the apple-grower and bee-keeper's two production activities exploit a natural symbiosis. Public goods, on the other hand, apart from being more usually thought of as goods rather than bads such as damage to the environment, are typically provided by public authorities. Yet, as remarked in Section 9.2, publicly owned facilities, including hospitals and even prisons, can be and in some cases actually are administered by private firms. And, of course, private individuals are hired to work in any civil service or government bureaucracy.

These differences are really not very important, however, at least when it comes to thinking about how to describe a complete economic allocation, including all the externalities and public goods which are created. For this reason, the rest of this chapter will not distinguish between the two — a common theoretical framework will be used for both. Indeed, an externality will be treated as an aspect of the public environment, in effect. The only novel feature will be that externalities will result from individual actions by consumers and producers, as opposed to much of the standard theory which concentrates upon public goods that are created by public authorities.

Now, externalities involve individual agents in two different ways. First, individual consumers and firms create externalities. Even very harmful externalities are not created out of sheer perversity; noisy parties are enjoyed by the participants, noisy motor cycles by their riders, firms create pollution — even extremely noxious nuclear waste — because

to avoid doing so would require costly resources and lower their profits. This aspect of externalities should of course be captured in the description. So, of course, must the impact of externalities on individual consumers and firms. Here, since most externalities affect indiscriminately whoever happens to be near their source, it is natural to treat the impact as though the externality were a public "bad".

Both public goods and externalities will be described by members of some *externality set* E, so that an *externality vector* will typically be taken as a member of the finite-dimensional Euclidean space $\Re^E$. Any $e \in \Re^E$ could be thought of as a vector of non-traded commodities, such as air or water quality, or as a vector of activities which create externalities.

Consider any individual consumer $i \in I$. Write $e^i \in \Re^E$ for the externalities which i creates, and $z \in \Re^E$ for the externalities and the public environment i which experiences in common with anybody else. Then i's set of feasible net trades X^i will typically depend upon both e^i and z. For a consumer who creates more external diseconomies is thereby enabled to have more consumption opportunities, since such constraints as not exceeding certain noise limits are relaxed. On the other hand, increasing an external diseconomy component of z is liable to affect i adversely. For example, in order to sleep peacefully near a noisier airport, better sound insulation in the house is usually required. In the case of external economies, such as living near a beautiful park in which it is easy to go for an enjoyable walk, the direction of dependence is changed, but there will still be such dependence. Thus i's feasible set should be written as $X^i(e^i, z)$. Consumer i also has a preference ordering R^i defined over the set of all triples (x^i, e^i, z) satisfying $x^i \in X^i(e^i, z)$.

Private producers also both create externalities and are affected by them. Write d^j for the externalities which firm j creates, and z for the common externalities which firm j experiences along with everybody else. Producer j's feasible net output vectors y^j are the members of a production set $Y^j(d^j, z)$ which depends on both d^j and z. If some component of d^j increases, this may signify that the firm finds itself emitting more smoke, for instance, and this usually increases the firm's opportunities to increase its outputs or reduce its inputs. On the other hand, increasing one component of z may affect j beneficially or adversely, depending upon whether the component corresponds to an external economy or diseconomy.

Finally, the externality creating activities of all private agents together are represented by the list $(\mathbf{e}^I, \mathbf{d}^J) := ((e^i)_{i \in I}, (d^j)_{j \in J})$. There is an obvious link between $(\mathbf{e}^I, \mathbf{d}^J)$ and the common externality z affecting all private agents. I am going to assume that $z = \sum_i e^i + \sum_j d^j$. This is actually quite plausible; it really just involves measuring both the externality creation and the externality impact appropriately. If one component of z deals with a particular kind of air pollutant, for instance, then the total quantity of pollutant which affects everybody in a given locality is simply the sum of the quantities of that pollutant which individual agents release into the air in that locality. Of course, while this description allows the possibility that each agent both creates and is affected by externalities of every type, it is by no means required or expected that more than a very few agents will either create or be affected by any one type of externality. This is especially true if we remember that externalities, like all other commodities, should be distinguished by geographical location.

With this description of externalities, an *allocation* (with externalities) is a combination $(\mathbf{x}^I, \mathbf{y}^J, \mathbf{e}^I, \mathbf{d}^J, z)$ satisfying the feasibility constraints:

(1) $x^i \in X^i(e^i, z)$ for each $i \in I$;

(2) $y^j \in Y^j(d^j, z)$ for each $j \in J$;

(3) $\sum_i x^i = \sum_j y^j$ and $z = \sum_i e^i + \sum_j d^j$.

An allocation with externalities is *Pareto efficient*, of course, if there is no alternative allocation with externalities which makes one consumer better off without making another consumer worse off. The definition of weak Pareto efficiency is very similar — it must not be possible to make all consumers better off simultaneously.

10. Lindahl–Pigou competitive allocations

10.1. Lindahl prices and Pigovian taxes or subsidies

The remedy that standard theory suggests for creating such externalities is to tax agents who create them as a way of restoring the full Pareto efficiency of market allocations. This idea goes back to Pigou's *Economics of Welfare*. In addition, it will turn out that all the agents who are affected by an externality should receive compensation for it at different rates determined by how much marginal damage they suffer. This is the counterpart of the standard approach to the problem of achieving Pareto efficiency with public goods, named after Lindahl (1919), whereby all consumers and private producers are required to pay a "Lindahl price" for each public good equal to the marginal benefit that they receive from it. Then each public good is produced so that the total marginal benefit to all private consumers and producers is equal to the marginal cost of providing it. Here any public goods will be included with other aspects of the environment. Thus, in the case of negative externalities or "external diseconomies" the government is assumed to collect taxes on their creation, and to remit this tax revenue as compensation to those agents who suffer damage. But for the reverse case of positive externalities or "external economies" the government is expected to subsidize their creation, and to finance the cost of doing so by charges upon those agents who benefit from the externalities. In the discussion below, I shall save space by speaking only of external diseconomies, but of course the theory can also accommodate external economies by means of a simple change of sign.

By having both taxes on externality creation and compensation for damage caused, it becomes possible to charge each agent at exactly the same rate for the same amount of externality creation. This, of course, is entirely sensible; a given amount of smoke released into the air at a given location causes the same damage no matter who releases it, so the charge should be the same if one is to achieve Pareto efficiency by restoring the balance between social and private benefits and costs. Some agents are less likely to emit smoke because they themselves would be harmed by it. This will be allowed for in the compensation payments.

So, in addition to a commodity price vector p, there are also *Pigovian taxes* $t \in \Re^E$ on externality creation, and a profile of *Lindahl price vectors* $((q^i)_{i \in I}, (q^j)_{j \in J}) =: \mathbf{q}$ indicating at what rate compensation should be paid to agents affected by externalities. Each Lindahl

price vector q^i $(i \in I)$ and q^j $(j \in J)$ should also be a member of $\Re^E$, so that there is one separate component for each kind of externality or public good. Given this system of prices, taxes, and compensation payments, consumers maximize their respective preference orderings subject to the relevant budget constraint, and producers maximize their profits. Of course, each agent chooses a vector of net trades and creates a vector of externalities. But in addition the compensation payments are arranged so that agents also unanimously demand to experience the exact total externality which actually affects them all in equilibrium.

This leads to the definition of a *Lindahl–Pigou competitive allocation* $(\hat{\mathbf{x}}^I, \hat{\mathbf{y}}^J, \hat{\mathbf{e}}^I, \hat{\mathbf{d}}^J, \hat{z})$ at *prices p*, *taxes t*, and *marginal compensation payments* $\mathbf{q}$ as an allocation which satisfies all the following conditions:

(1) for all $i \in I$, one has $\hat{x}^i \in X^i(\hat{e}^i, \hat{z})$, and if $x^i \in X^i(e^i, z)$ with $p\,x^i + t\,e^i - q^i z \leq p\,\hat{x}^i + t\,\hat{e}^i - q^i\hat{z}$, then $(\hat{x}^i, \hat{e}^i, \hat{z})\ R^i\ (x^i, e^i, z)$;

(2) for all $j \in J$, one has $\hat{y}^j \in Y^j(\hat{d}^j, \hat{z})$, and if $y^j \in Y^j(d^j, z)$ then $p\,y^j - t\,d^j + q^j z \leq p\,\hat{y}^j - t\,\hat{d}^j + q^j\,\hat{z}$;

(3) $t = \sum_i q^i + \sum_j q^j$;

(4) $\sum_i \hat{x}^i = \sum_j \hat{y}^j$ and $\hat{z} = \sum_i \hat{e}^i + \sum_j \hat{d}^j$.

Thus (1) requires each consumer $i \in I$ to choose a net demand vector x^i, levels of externality creation e^i, and levels of damage z, in order to maximize the preference ordering R^i subject to a budget constraint which includes both taxes on externality creation and compensation for damage. Next, (2) requires each producer $j \in J$ to choose a net output vector y^j, levels of externality creation d^j, and levels of damage z, in order to maximize profit — taking as given the system of prices, taxes, and compensation payments. By (3), tax rates must be set equal to total marginal damage, whereas (4) requires market clearing for both commodities and externalities, without assuming free disposal. Notice that the total taxes collected are $t\,\hat{z}$, while the total compensation paid to all agents is $\sum_i q^i\hat{z} + \sum_j q^j\,\hat{z}$, which is equal to $t\,\hat{z}$ because of (3). So the public sector budget constraint is automatically satisfied; indeed, for each separate externality (or public good), what the government collects as tax revenue (or pays out in subsidies to creators of the public good) is exactly equal to what it pays as compensation (or receives from Lindahl charges for the public good).

Notice also that q^i and q^j really do represent the vector of marginal rates of damage which the various externalities cause to private agents. For any type of externality s, the

Lindahl price $(-q_s^i)$ represents i's marginal rate of substitution between that externality and the *numéraire* commodity; $(-q_s^j)$ represents the (negative) marginal profit which firm j earns from increasing that externality.

10.2. An equivalent private goods economy

It is now possible to show that Lindahl–Pigou competitive allocations of private goods and of externalities or public goods altogether have exactly the same efficiency properties as competitive allocations of private goods alone. That is, the two efficiency theorems which were presented in Sections 6 and 8 above remain true. Moreover, they have a simple proof. For, instead of having to verify them by checking that essentially all the same proofs still work, it is enough to show instead that the economy with private goods and externalities now being considered is equivalent to an economy with only private goods of the kind that was discussed in the earlier part of this chapter.

The equivalence relies on adapting an idea first set out formally by Milleron (1972, p. 428) — see also Foley (1970). There will be a separate *individualized* public environment z^i for each consumer $i \in I$, and also a separate z^j for each producer $j \in J$. Of course, since no person is an island, but has to share the same earth and so the same public environment with everybody else, physical feasibility entails having $z = z^i = z^j$ for each consumer $i \in I$ and for each producer $j \in J$. So the set J will be extended to include an extra fictitious producer labelled 0, whose production set will constrain feasible allocations in the whole economy so that they must satisfy these equations. Then the set of producers becomes equal to $J \cup \{0\}$, while the set of commodities becomes equal to $G \cup E \cup [E \times (I \cup J)]$, and so the commodity space becomes $\Re^{G \cup E \cup [E \times (I \cup J)]}$.

In this fictitious private good economy, for any pair of agents (consumers or producers) $a, b \in I \cup J$, the vector z^{ab} denotes agent a's net trade in agent b's individualized public environment. In fact it will not be feasible for any agent to trade in the individualized public environment of any other agent. Accordingly, each consumer $i \in I$ has an "extended" feasible set:

$$\tilde{X}^i = \{\, (x^i, e^i, z^{iI}, z^{iJ}) \in \Re^{G \cup E \cup [E \times (I \cup J)]} \mid x^i \in X^i(e^i, z^{ii});$$
$$z^{ih} = z^{ij} = 0 \ (\text{all } h \in I \setminus \{i\}; \text{all } j \in J) \,\}.$$

Moreover, each consumer $i \in I$ has an "extended" weak preference ordering $\tilde{R}^i$ over the set $\tilde{X}^i$ satisfying:

$$(x^i, e^i, z^{iI}, z^{iJ}) \; \tilde{R}^i \; (\bar{x}^i, \bar{e}^i, \bar{z}^{iI}, \bar{z}^{iJ}) \iff (x^i, e^i, z^{ii}) \; R^i \; (\bar{x}^i, \bar{e}^i, \bar{z}^{ii})$$

Similarly, each producer $j \in J$ has an "extended" feasible set:

$$\tilde{Y}^j = \{ (y^j, d^j, z^{jI}, z^{jJ}) \in \Re^{G \cup E \cup [E \times (I \cup J)]} \mid y^j \in Y^j(d^j, z^{jj});$$
$$z^{ji} = z^{jk} = 0 \text{ (all } i \in I; \text{ all } k \in J \setminus \{j\}) \, \}.$$

Finally, the fictitious extra producer 0 will be given the feasible set:

$$\tilde{Y}^0 = \{ (y^0, e^0, z^{0I}, z^{0J}) \in \Re^{G \cup E \cup [E \times (I \cup J)]} \mid y^0 = 0; \; e^0 = z^{0i} = z^{0j} \text{ (all } i \in I; \text{ all } j \in J) \, \}.$$

Then a feasible allocation is a combination

$$(\mathbf{x}^I, \mathbf{e}^I, \mathbf{z}^{I \times (I \cup J)}, \mathbf{y}^J, \mathbf{d}^J, \mathbf{z}^{J \times (I \cup J)}, y^0, e^0, z^{0I}, z^{0J})$$

of vectors in this commodity space, one for each agent in the economy, satisfying the feasibility constraints:

(1) $(x^i, e^i, z^{iI}, z^{iJ}) \in \tilde{X}^i$ for each consumer $i \in I$;

(2) $(y^j, d^j, z^{jI}, z^{jJ}) \in \tilde{Y}^j$ for each producer $j \in J$;

(3) $(y^0, e^0, z^{0I}, z^{0J}) \in \tilde{Y}^0$ for the fictitious producer 0;

(4) $\sum_j y^j - \sum_i x^i = 0$, $\sum_i e^i + \sum_j d^j = e^0$, $z^{ii} = z^{0i}$ (all $i \in I$), and $z^{jj} = z^{0j}$ (all $j \in J$).

Of these, the first three are obvious individual feasibility conditions. The fourth is a collection of resource balance constraints for each possible commodity, including the individualized public environment of each agent. They allow producer 0 to be regarded as making demands e^0 for externalities which exactly balance the total supplies of all externalities by all consumers and "real" producers in the economy. In addition, producer 0 matches the individualized public environment which each agent demands. The equalities $e^0 = z^{0i} = z^{0j}$ which are built into the definition of $\tilde{Y}^0$ then ensure that an allocation is feasible only if $\sum_i e^i + \sum_j d^j = z^{ii} = z^{jj}$ (all $i \in I$; all $j \in J$). But this is exactly the condition used above in the original definition of a feasible allocation with public goods and externalities.

It remains only to check that the usual definition of a competitive allocation in this private good economy with an individualized public environment for each agent is equivalent

to the definition given above of a Lindahl–Pigou competitive allocation for an economy with private goods, public goods, and externalities. In order to do this, note that the typical price system will also be a vector in $\Re^{G \cup E \cup [E \times (I \cup J)]}$. This may as well be written in the form $(p, t, \mathbf{q}^I, \mathbf{q}^J)$ where p is an ordinary commodity price vector in $\Re^G$, t is a Pigou tax vector of charges for creating externalities that lies in $\Re^E$, while $(\mathbf{q}^I, \mathbf{q}^J)$ is a collection of Lindahl price vectors for each agent's personalized public environment that lies in $\Re^{E \times (I \cup J)}$. For each consumer $i \in I$, because any $(x^i, e^i, z^{iI}, z^{iJ}) \in \tilde{X}^i$ must satisfy the equations $z^{ih} = z^{ij} = 0$ (all $h \in I \setminus \{i\}$; all $j \in J$) which are built into the definition of $\tilde{X}^i$, the budget constraint for consumer $i \in I$ can be simplified to $p\,x^i + t\,e^i - q^i\,z^{ii} \leq w^i$ where w^i denotes i's unearned wealth. Similarly, the expression for the profits of each producer $j \in J$ can be simply expressed as $p\,y^j - t\,d^j + q^j\,z^{jj}$. Finally, the fictitious producer 0 produces under constant returns to scale, so that an allocation can only be competitive if producer 0 is unable to earn profits. Yet producer 0's profits from choosing $e^0 = z^{0i} = z^{0j}$ (all $i \in I$; all $j \in J$) and supplying (z^{0I}, z^{0J}) while demanding e^0 are $\left(\sum_i q^i + \sum_j q^j - t \right) e^0$. These will be zero no matter what producer 0 chooses if and only if $t = \sum_i q^i + \sum_j q^j$. Hence this is a condition that any Lindahl–Pigou price system must satisfy.

Having suitably formulated the price system, each consumer's budget constraint, and each producer's profits, it is then routine to check that preference and profit maximization, etc., are satisfied in an appropriate way. So the efficiency theorems of Sections 6 and 8 really do carry over to an economy with public goods and externalities — provided, of course, that the the assumptions of Section 8 in particular are satisfied in this equivalent private good economy. So, of course, do the existence theorems and some results concerning the core which are presented in other chapters of this book. However, the core does *not* shrink to the set of Lindahl–Pigou equilibrium allocations as the economy is replicated, nor is there core equivalence in the limit (see Muench, 1972). This is because, when there are public goods, the dimension of the relevant commodity space expands in proportion to the number of individuals.

11. Instances of Market Failure

11.1. Introduction

Sections 9 and 10 have shown that public goods and externalities, which are often blamed for the failure of markets to achieve Pareto efficient allocations, in fact do not create any obstacles to efficiency by themselves. Now it is time to consider other ways in which markets can fail. Relatively mild forms of market failure arise when equilibrium is Pareto inefficient because some goods are not traded, or when the market itself reaches an equilibrium which is not a Walrasian equilibrium, because some firms or other traders behave monopolistically, and do not take prices as fixed and given. The latter problem of monopoly power is in principle very important. But it probably lies beyond the scope of this book, and certainly beyond that of this chapter. So let me pass on straight away.

As for missing markets, they are often thought to arise in connection with public goods and externalities. Yet what Sections 9 and 10 showed for them is true more generally. Until we have at least considered whether there may not be more fundamental reasons for markets to be incomplete, it really is inappropriate to blame missing markets as an apparent source of Pareto inefficiency. After all, the results presented in Sections 9 and 10 teach us that public goods and externalities by themselves do not stand in the way of completing the market system by means of a Lindahl–Pigou pricing scheme and so of achieving Pareto efficiency. There can even be a complete market system with a just distribution of wealth. It appears, in fact, that any problems created by public goods and externalities are actually more of a symptom than a cause of those Pareto inefficiencies which can be ascribed to missing markets. Accordingly, the rest of this chapter will seek explanations for market incompleteness, as well as for other sources of market failure.

Indeed, the problems with the market mechanism as a way of allocating resources and with the above efficiency theorems arise once one begins to think seriously about the realism of the assumptions underlying those results. This is especially true of the second efficiency theorem presented in Section 8. Unfortunately for the theory, this is the only theorem that could be considered as having any practical interest, unless the existing maldistribution of wealth could somehow be regarded as ethically justified. Even the first theorem, however, will only apply to economies where complete competitive market equilibrium is guaranteed as a theoretical possibility.

11.2. Non-convexities and non-existence

Accordingly, the most drastic kind of market failure arises when markets in equilibrium cannot be used to realize a Pareto efficient allocation because no Walrasian equilibrium exists at all. There are several possible reasons why this might happen, but not all these reasons are equally plausible. For example, in Section 2 most of the assumptions concerning individual consumers' preferences and feasible sets happen to ensure that their demands all respond continuously to changes in prices. Yet existence of Walrasian equilibrium really only requires consumers *in aggregate* to have continuous demand responses to price changes. In real economies where there are very many consumers, the average demand per consumer will tend to be a continuous function of prices even for goods which are essentially indivisible. For example, each consumer either does or does not purchase a particular model car in a given year, and very few indeed buy more than one new car in a year. Clearly, each individual consumer has an essentially non-convex feasible set. But even with only one million consumers, the number of cars bought per consumer becomes practically indistinguishable from a variable which varies continuously with prices.[8] So the assumption that consumers have convex preferences, though implausible if taken literally, is not really necessary if there are very many consumers.

In fact the main obstacles to existence of Walrasian equilibrium allocations usually come about on the production side of the economy. The key assumption of Section 3 was that the aggregate production set of the economy is convex. Unlike for consumers, however, it is *not* plausible to assume that there is such a large number of small firms in every industry that the supply of each good is effectively continuous. Economies of scale in industries such as chemicals, steel-making, and electronics imply that it is inefficient to have more than a small number of very large plants. Actually, in some industries such as aircraft manufacturing or the production of silicon chips, it is apparently becoming inefficient to have more than a few plants in the whole world to manufacture products for one world market.

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[8] A completely rigorous statement of this idea that demands become continuous as the number of consumers becomes large inevitably involves several technical difficulties. See Hildenbrand (1974) and Trockel (1984) for technically accomplished presentations, and Farrell (1959) for a more elementary and intuitive approach. For a discussion of the second efficiency theorem with non-convex preferences, see Anderson (1988).

C (total cost
   of pollution)

$\bar{C}$

C(e)

0

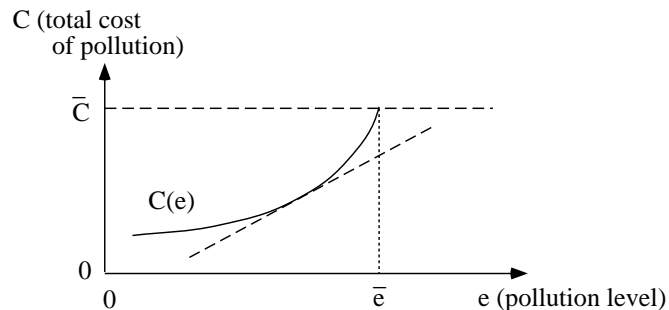0 $\bar{e}$ e (pollution level)

Figure 9

As Starrett (1972) has pointed out, moreover, it is almost inevitable that negative externalities will give rise to fundamental non-convexities, and so limit the applicability of the Lindahl–Pigou pricing scheme. The reason is astonishingly simple. Consider first the case of a producer whose profit opportunities are being damaged by some form of pollution. Convexity requires those opportunities to be described by means of a convex extended production possibility set which includes the level of external pollution e as one (negative) component of the firm's net output vector. In particular C, the total cost of pollution to the producer, must be a convex function $C(e)$ of e, implying that marginal cost $C'(e)$ must be non-decreasing in e. So, as Fig. 9 illustrates, the pollution total cost curve must lie on or above any line which its tangent at some point. Since the marginal cost is presumably positive for at least some levels of pollution, any such tangent must have a positive slope for higher levels of e. It follows that $C(e)$ must increase without bound as e becomes indefinitely large. Eventually this cost must be large enough to overwhelm any profit which the firm may be able to make from its normal production activities. So the firm will prefer to earn a zero profit by staying out of business entirely if e exceeds some critical finite value $\bar{e}$ at which costs rise to some critical level $\bar{C}$. When e is already so high that the firm refuses to operate, however, extra pollution cannot harm it any further. Thus $C(e) = \bar{C}$ and $C'(e) = 0$ for all $e > \bar{e}$. It follows that the marginal cost of pollution must somehow decrease from positive levels to zero as e becomes large. Yet this is incompatible with a convex production possibility set.[9]

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[9] For further analysis, see Otani and Sicilian (1977).

While such fundamental non-convexities are more obvious for producers, they can easily arise in consumers' feasible sets as well. For suppose that a consumer's ability to supply some kind of labour service is adversely affected by the relevant form of pollution, perhaps because it causes severe breathing difficulties, and that very high levels of pollution will permanently disable or even kill the consumer. Then the same diagram as Fig. 9 applies, except that C should be re-interpreted as lost potential earnings from working. Convexity of the consumer's opportunities requires that $C(e)$ should be a convex function, yet there will be a critical value $\bar{e}$ at which the consumer's earning opportunities have fallen to zero and beyond which marginal damage must also be zero. Of course, this does not rule out the likelihood that more pollution will add to the consumer's suffering, but once lethal levels are reached even this is no longer true. External diseconomies really are associated with fundamental non-convexities.

It was argued above that non-convexities might not be so serious on the consumer side of the economy because consumers are so numerous that in aggregate they will come close to satisfying the relevant convexity assumptions. This is true in an economy with only private goods. With externalities and public goods, however, the equivalent private goods economy of Section 10.2 involves personalized copies of the public environment for each separate individual. Then, as the number of consumers increases, so does the dimension of the relevant commodity space. It is not enough to look at the average of all consumers' demands for a public good; instead, the entire interpersonal distribution of demands for personalized copies of that public good has to be considered, and this distribution has no obvious convexity properties that result from aggregating over many individuals.

The problem of non-convexities in production and in connection with negative externalities therefore merits serious discussion. Yet there is no easy way to overcome the failure of pricing schemes to reach a Pareto efficient allocation even in as simple an economy as that illustrated in Fig. 5 of Section 7. It really seems necessary to go beyond ordinary markets and impose some sort of direct quantity controls. For example, a firm such as the one in that example whose fixed costs are too high to allow it to earn a profit at the prices which consumers are willing to pay, even at the desired Pareto efficient allocation, could be required to incur those fixed costs while receiving some kind of production subsidy to help meet its financial obligations. Once such quantitative controls are admitted, however, it seems easy

in principle to reach an efficient allocation directly. Wherever markets fail, governments step in and institute direct controls to ensure that Pareto efficiency is restored. Needless to say, such perfect intervention is not seen in practice, nor is it even really practicable. This, however, probably has much more to do with transactions costs, limited information, and similar severe obstacles to Pareto efficiency, rather than with non-convexities *per se*. This takes us to the next two kinds of market failure, which will not be so easy to overcome.

11.3. *Physical transactions costs*

Organizing real markets takes real resources. The sellers have to work, and the buyers must put in some time even if they regard shopping as closer to an enjoyable leisure activity than to a form of unpaid labour. Transport is needed to bring goods to market, and to take purchases away for later consumption or use. Even those modern financial markets in which most transactions take place electronically require large investments in computing resources, as well as human operators to initiate and oversee various transactions. Such physical transactions costs place obvious limits on the number and extent of markets which can and do function in the economy. Markets are very unlikely to be complete after all, because administering a complete system of markets would be unreasonably expensive. Indeed, complete markets really require agents to be able to make forward purchases and sales that cover all their lifetime needs in every possible future contingency. Even more, they also really require agents to be able to make transactions for all the lifetime needs of their as yet unborn potential children, grandchildren, great-grandchildren, etc. This is clearly absurd, and the only practical way of organizing markets is probably rather close to what we see in reality, with many markets for goods which will be delivered almost immediately, but only a very few involving goods or promises of future payments which will not be delivered until after a remote future event occurs. The very uncertain benefits of setting up markets to determine what will happen only in remote future events seem greatly outweighed by the certain costs of organizing them now.

Once it is realized that transactions costs evidently limit the markets that can and will function in a real economy, it is very tempting to conclude that the resulting allocations will not be Pareto efficient. In the usual sense of Pareto efficiency that has been used up to now, moreover, this tempting conclusion is nearly always accurate. But this sense may not be the most appropriate in the presence of transactions costs. After all, the dividing

line between Pareto efficient and inefficient allocations will be most useful if it gives us a signal of when intervention in the market system would be desirable. It is nice to know that Pareto efficiency implies that the only grounds for intervention are distributional, in order to alleviate poverty even if that can only be done by making some rich people worse off. And it is even nicer to know that Pareto inefficiency really does mean that some Pareto improvement to the economic system is possible. This second statement becomes highly dubious in the presence of transactions costs, however. For the definition of efficiency in Section 4 presumes that the only limits on the set of feasible allocations are the usual individual feasibility and resource balance constraints. Nothing is said about the cost of transacting on markets — i.e., about precisely those costs that cause markets to be incomplete and so lead to the alleged inefficiency. Yet those costs may be very hard to circumvent even if the policy intervention which is designed to reach a Pareto superior allocation is very judicious indeed.

In other words, there are almost certain to be some transactions costs no matter what economic system is used in the actual allocation of goods and services. Because of this extra limitation on what allocations are really feasible, we should only call an allocation Pareto inefficient if there is some alternative economic system which, even after covering any relevant transactions costs, can still produce Pareto superior allocations. This is a much harder test to meet than the one that ignores transactions costs, and it may well be the case that even an incomplete market system meets it. Indeed, it is very likely that a complete market system would *not* meet it, but would fail catastrophically instead by absorbing almost all the world's resources in the transactions costs necessary to run the complete market system.

This discussion shows the need for a more refined notion of Pareto efficiency that can allow for transactions costs. What is required, in fact, is a notion of *constrained Pareto efficiency*, meaning that a smaller set of feasible allocations is considered. In particular, for an allocation to be regarded as feasible, it is necessary to exhibit a possible economic system which can really achieve that allocation, even after allowing for the transaction costs which prevent markets from being complete. Then an allocation resulting from incomplete markets can be called (constrained) Pareto inefficient only if there is a Pareto superior allocation in this constrained feasible set, and so only if it really is possible to arrange that such an allocation results from a new economic system allowing intervention in markets.

Surprisingly little research has yet been done on seeing when incomplete markets with transactions costs can pass the less stringent test of constrained Pareto efficiency. One reason may be that there are some serious conceptual difficulties which certainly go beyond the scope of this chapter. Nevertheless, to the extent that nobody has yet been able to suggest in a systematic way how to make Pareto improvements to equilibrium allocations in markets that are incomplete because of transactions costs, it seems unreasonable to say that markets fail because of transactions costs. They may fail to produce "first best" Pareto efficient allocations because, in the absence of transactions costs, there may be Pareto superior allocations. Yet most, if not all, of these alternative allocations may not really be feasible because any other economic system would face similar transactions costs.

Apart from incomplete markets, transactions costs can also help to explain some other apparent sources of inefficiency in actual economic systems. One reason is that, although lump-sum redistribution of wealth together with complete Lindahl–Pigou pricing for all public goods and externalities can often be used in theory to achieve a desirable Pareto efficient allocation, such procedures are likely to incur excessive transaction costs in practice. It may well be possible to lower transaction costs by instituting commodity and income taxes to finance expenditure on public goods, as well as on wealth transfers which are needed to improve the distributive justice of the economic system. Though such taxes appear to "distort" the market system and to introduce obvious Pareto inefficiencies, in fact it may be impossible to arrange any of the Pareto improvements that exist in theory because of the transactions costs that arise in practice. Thus, transactions costs do much to undermine the practical significance of the second efficiency theorem because many constrained Pareto efficient allocations are likely to rely on introducing distortions into a market system in order to economize on transactions costs. Even more troubling, perhaps, is the likelihood that market forces may actually make the allocation worse because they encourage individual consumers and producers to try to avoid paying taxes or charges for creating undesirable externalities, and also to circumvent direct controls on transactions such as absolute prohibitions on buying undesirable goods. In other words, market forces may add to transactions costs because suppressing them would be difficult. Very similar problems arise when there are no physical transactions costs, but when limited information causes similar obstacles for an efficiently functioning economic system, as discussed in Hammond (1979, 1987, 1990).

11.4. Limited information

Physical transactions costs, as we have just seen, do help to explain why markets are incomplete, and why distortionary taxes play such an essential role in most economic systems. But limited information puts even more powerful constraints on what an economic system can achieve, as well as on what markets can function. It is as if limited information introduces a form of "informational" transactions cost to add to any physical transactions costs. After all, when information is private, the true demand and supply functions of individual consumers and producers are not known. This gives them the opportunity to act monopolistically and to manipulate market prices, usually by understating their true willingness to trade. Similarly, the true skills of workers are unknown. This makes it impossible to base lump-sum redistribution of wealth on individuals' true earning capacities, as would be theoretically ideal. Instead, taxes have to be levied on actual income rather than on a proper assessment of earning potential. Also, in making transfers to the poor it will be difficult to distinguish between the genuinely needy and those who wish to exploit whatever system of poverty relief is created. Finally, each agent's true willingness to pay for public goods or for an improved environment is also unknown. If Lindahl pricing is used to determine what public environment to produce and also who should pay for it, limited information will give rise to the "free-rider problem." Individual agents will each benefit by "free-riding" — i.e., by understating their true willingness to pay so that they pay less personally, even though such behaviour reduces the total supply of any public good, and also leaves the state of the environment worse than individuals really want and are willing to pay for.

In the past such limited information has often been seen by economists such as Hayek and others as creating an opportunity for markets. It is argued that central authorities cannot function properly, but that one needs a decentralized economic system such as markets appear to provide, with everybody relying only on what they themselves know, and not on knowledge about anybody else. Carried to its extreme, this argument suggests that there should be no intervention at all in the market system, but only complete *laissez faire.* Yet this overlooks the important fact that limited information on its own can cause serious failures even in perfectly competitive markets, as was originally pointed out in the important papers by Arrow (1963), Akerlof (1970), and in much subsequent work by many

other authors — see, for example, Stiglitz (1987). Even more evidently, complete *laissez faire* is also likely to leave us without any public goods, including controls on air pollution, etc. And, with no welfare programs to alleviate poverty, or even to take care of prisoners, a horrific and crime ridden society is likely to be the result.

A truer picture of the role of markets is inevitably much more complicated. Limited information makes it more costly, but not entirely impossible, for central authorities to function. It also increases the need for such authorities, however. Ultimately, there has to be a trade-off between the undesirable aspects of complete *laissez faire* discussed above, and the reduction in administrative and transactions costs that decentralized markets can often bring about. This, however, is an entirely different understanding of markets than that suggested by the analysis in the first part of this chapter.

12. Conclusions

This chapter has expounded the efficiency properties of complete and perfectly competitive markets. The central results are the two efficiency theorems set out and proved in Sections 6 and 8 above. The first theorem says that such competitive markets produce Pareto efficient allocations — though only weakly Pareto efficient, in general, if some consumers happen to have locally satiated preferences. There is no guarantee of anything like distributive justice, however. The second theorem may therefore be more interesting, since it says that any Pareto efficient allocation, distributively just or unjust, can be supported by means of complete and perfectly competitive markets in combination with lump-sum redistribution of wealth. The second theorem, however, is only valid under rather restrictive assumptions. These are that consumers have continuous, convex, and locally non-satiated preferences, that the aggregate production set of all producers is convex, and that (as explained in Section 8) all commodities are relevant. Moreover, the efficient allocation in question should not require such an extreme distribution of wealth that it is "oligarchic".

After presenting these two main theorems, the last three Sections have been concerned with sources of "market failure". Though public goods and externalities are often blamed for the inefficiency of a market system, Sections 9 and 10 argued that it might be better to see them as symptoms rather than causes of missing markets. Finally, Section 11 discussed rather briefly what seems to be the most important cause of market failure — namely,

physical transactions costs, and also the "informational" transactions costs which limit what an economic system can achieve when it must cope with limited information. When there are such transactions costs, (constrained) Pareto efficiency may require intervention in markets by means of distortionary taxes or even direct controls of some kind. Then market forces can actually add to the problems of creating a good economic system by encouraging tax evasion, black markets, or other ways of reducing the effectiveness of interventionary policies.

As is often the case in economic theory, we have some very powerful results, but we must also resist being tempted to overestimate their practical significance.

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