1

$$\max_{c_s, \tilde{c}_{s+1}, \ldots, \tilde{c}_{T-1}} \mathbb{E}_s \left[\sum_{t=s}^{T-1} \beta_t u(c_t) \,\middle|\, w_s \right] = \mathbb{E}_s U_s(\tilde{c})$$

$u' > 0$, $u'' < 0$.

$\cancel{w_{t+1} = w_t - c_t + \tilde{r}_t}$.

risks $\tilde{r}_t$; i.i.d. independent (identically) distributed.

$w_{t+1} = \tilde{r}_t (w_t - c_t(w_t))$ w_s given.

Know w_t at ~~time~~ t, but not before.

(w_t depends on $\ldots r_{t-2}, r_{t-1}$).

Terminal condition $w_T \geq 0$.

$$\beta_s u(c_s) + \mathbb{E}_s \left[\beta_{s+1} u(c_{s+1}(w_{s+1})) + \mathbb{E}_{s+1} [\beta_{s+2} \ldots \right.$$

Remember $\mathbb{E}_s \mathbb{E}_{s+1} = \mathbb{E}_s$.

iterated expectations

$\cancel{\mathbb{E}_s \mathbb{E}_{s+1} g = \int_s}$

$$\cancel{\mathbb{E}_s \mathbb{E}_{s+1} g(w_s, w_{s+1}) = \int_{\Omega_s \times \Omega_{s+1}} g(w_s, w_{s+1}) f(w_s, w_{s+1})\, dw_s\, dw_{s+1}}$$

$$\cancel{= \int_{\Omega_s} f_s(w_s) \left(\mathbb{E}_{s+1} g(w_s, w_{s+1}) \right) dw_s}$$

②

$$\mathbb{E}_s\, g(\omega_s, \omega_{s+1})$$

$$= \mathbb{E}_s\left[\mathbb{E}_{s+1}\left[g(\omega_s, \omega_{s+1}) \mid \omega_s\right]\right]$$

$$\text{Max } \mathbb{E}_s \sum_{t=s}^{T} \beta_t u(c_t)$$ ω_s fixed,

ST $\omega_{t+1} = \tilde{r}_t(\omega_t - c_t)$, $\omega_{T+1} \geq 0$.

~~[illegible]~~ Work backwards.

max value
state function

In period T, (when $s = T$),

$c_T = \omega_T$, $V_T(\omega_T) = \beta_T u(\omega_T)$.

In period $T-1$, $s = T-1$,

$$\max_{c_{T-1}} \left\{\beta_{T-1} u(c_{T-1}) + \mathbb{E}\, V_T(\omega_T)\right\}$$

period $T-1$.

result of an optimal policy in period T.

$$\omega_T = \tilde{r}_{T-1}(\omega_{T-1} - c_{T-1})$$

random gross return

saving

$$\max_{c_{T-1}} \beta_{T-1} u(c_{T-1}) + \mathbb{E}\, \beta_T u(\tilde{r}_{T-1}(\omega_{T-1} - c_{T-1})).$$

$$c_{T-1} \leq \omega_{T-1}$$

$$0 = \beta_{T-1} u'(c_{T-1}) - \mathbb{E}\, \beta_T u'(\tilde{r}_{T-1}(\omega_{T-1} - c_{T-1}))\tilde{r}_{T-1}.$$

$$u'(c_{T-1}) = \mathbb{E}\, \frac{\beta_T}{\beta_{T-1}} \tilde{r}_{T-1} u'(\tilde{r}_{T-1}(\omega_{T-1} - c_{T-1})).$$

$$1 = (\beta_T / \beta_{T-1})\, \mathbb{E}\, \tilde{r}_{T-1} \left[u'(\tilde{r}_{T-1}(\omega_{T-1} - c_{T-1})) / u'(c_{T-1})\right].$$

Stochastic Euler equation.

$$u(c) = \frac{c^{1-\varepsilon}}{1-\varepsilon}, \quad (\varepsilon > 0)$$

ε risk and fluctuation aversion

(even if $\varepsilon = 1$)

$$\boxed{u'(c) = c^{-\varepsilon}}$$

$$c_{T-1}^{-\varepsilon} = \frac{\beta_T}{\beta_{T-1}} \mathbb{E}\, \tilde{r}_{T-1} \left[\tilde{r}_{T-1}(\omega_{T-1} - c_{T-1})\right]^{-\varepsilon}$$

$$= \frac{\beta_T}{\beta_{T-1}} (\omega_{T-1} - c_{T-1})^{-\varepsilon}\, \mathbb{E}\, \tilde{r}_{T-1}^{1-\varepsilon}$$

$$c_{T-1} = \left(\frac{\beta_T}{\beta_{T-1}}\right)^{-\frac{1}{\varepsilon}} (\omega_{T-1} - c_{T-1}) \left(\mathbb{E}\, \tilde{r}_{T-1}^{1-\varepsilon}\right)^{-\frac{1}{\varepsilon}}$$

$$c_{T-1} = \gamma_{T-1} \omega_{T-1}, \quad \omega_T = \tilde{r}_{T-1}(1 - \gamma_{T-1})\omega_{T-1}.$$

γ_{T-1} – ind. of ω_{T-1}

5

$$\cancel{\gamma_{T-1} = \frac{C_{T-1}}{w_{T-1}}}$$

$$\gamma_{T-1} = \left(\frac{\beta_T}{\beta_{T-1}}\right)^{-1/\varepsilon} (1-\gamma_{T-1}) \left(\mathbb{E}\, \tilde{r}_{T-1}^{\,1-\varepsilon}\right)^{-\frac{1}{\varepsilon}}$$

With $u(c) = \dfrac{c^{1-\varepsilon}}{1-\varepsilon}$ $(\varepsilon > 0)$.

$$V_s(w_s) = \max \mathbb{E}_s\left[\sum_{t=s}^{T} \beta_t u(c_t) \mid w_s\right]$$

with w_s given.

s.t. $w_{T+1} \geq 0$

$c_t = \gamma_t w_t$

$$= \alpha_s \frac{w_s^{1-\varepsilon}}{1-\varepsilon}.$$

$\alpha_s, \gamma_s > 0$.

constants (depend on probs?).

STATE VALUATION FUNCTION.

Induction hypothesis.

BELLMAN equation (finite horizon).

True for general u.

$$\left.\begin{aligned} V_s(w_s) &= \\ c_s(w_s) &= \arg \end{aligned}\right\} \max_{c_s} \left\{\beta_s u_s(c_s) + \mathbb{E}_s V_{s+1}(\hat{w}_{s+1})\right\}$$

cond. on w_s, c_s.

PRINCIPLE OF OPTIMALITY.

PROVE BY BACKWARD INDUCTION (when $u(c) = \dfrac{c^{1-\varepsilon}}{1-\varepsilon}$)

that $V_s(w_s) = \dfrac{\alpha_s w_s^{1-\varepsilon}}{1-\varepsilon}$, $c_s(w_s) = \gamma_s w_s$

PROOF. When $s = T$, $\gamma_T = 1$,

$$\alpha_T = \beta_T$$

Induction hypothesis: suppose true for $s+1$.

$$\left.\begin{array}{l} V_s(w_s) = \\ c_s(w_s) = \arg \end{array}\right\} \max_{\gamma_s} \left\{ \beta_s \frac{(\gamma_s w_s)^{1-\varepsilon}}{1-\varepsilon} + \mathbb{E}_s \frac{\alpha_{s+1}\left[\tilde{r}_s (1-\gamma_s) w_s\right]^{1-\varepsilon}}{1-\varepsilon} \right\}$$

$$\left.\begin{array}{l} \frac{\alpha_s}{1-\varepsilon} = \\ \gamma_s = \arg \end{array}\right\} \max_{\gamma_s} \left\{ \frac{\beta_s \gamma_s^{1-\varepsilon}}{1-\varepsilon} + \mathbb{E}_s \frac{\alpha_{s+1}\left[\tilde{r}_s (1-\gamma_s)\right]^{1-\varepsilon}}{1-\varepsilon} \right\}$$

maximand $\dfrac{\beta_s \gamma_s^{1-\varepsilon}}{1-\varepsilon} + \dfrac{\alpha_{s+1}(1-\gamma_s)^{1-\varepsilon}}{1-\varepsilon} \mathbb{E}_s \tilde{r}_s^{1-\varepsilon}$

(when returns are serially independent).

FOC: $\beta_s \gamma_s^{-\varepsilon} - \alpha_{s+1}(1-\gamma_s)^{-\varepsilon}\, \mathbb{E}\tilde{r}^{1-\varepsilon} = 0$

ind. of s.

$$\beta_s^{-1/\varepsilon} \gamma_s = \alpha_{s+1}^{-1/\varepsilon}(1-\gamma_s)\left(\mathbb{E}\tilde{r}^{1-\varepsilon}\right)^{-1/\varepsilon}.$$

$$0 < \gamma_s = \frac{\alpha_{s+1}^{-1/\varepsilon}\left(\mathbb{E}\tilde{r}^{1-\varepsilon}\right)^{-1/\varepsilon}}{\beta_s^{-1/\varepsilon} + \alpha_{s+1}^{-1/\varepsilon}\left(\mathbb{E}\tilde{r}^{1-\varepsilon}\right)^{-1/\varepsilon}} < 1.$$

$$\delta = \left(\mathbb{E}\tilde{r}^{1-\varepsilon}\right)^{-1/\varepsilon}$$

$\alpha_s = \ldots$ EXERCISE.

$$\max_{\underset{\sim}{y}} \mathbb{E}_s\left\{\sum_{t=s}^{T} u_s(\underset{\text{state}}{x_s}, \underset{\text{control}}{y_s}) + \phi_T(x_T)\right\}, \quad x_s \text{ given}$$

LAW OF MOTION

$$x_{t+1} = \xi_t(x_t, y_t, \varepsilon_t).$$

~~i.i.d. shocks.~~ Conditionally independent random shocks given x_t, y_t.

$$P_{t+1}(x_{t+1} \mid x_t, y_t).$$

conditional probs.

Controlled Markov process.

$$V_s(x_s) = \max_{y_s \in F_s(x_s)}$$

$$\boxed{y_s^*(x_s)} = \arg\max_{y_s \in F_s(x_s)}.$$

strategy

maximand

$$= u_s(x_s, y_s) + \mathbb{E}_s\left[V_{s+1}(x_{s+1}) \mid x_s, y_s\right]$$

immediate return

Bellman equation: Principle of optimality in dynamic ~~pro~~ programming.

"one-shot deviation principle" in games.

If any deviation improves, there exists a one-shot deviation that improves.

$$\max \sum_{t=s}^{\infty} \beta^{t-s} u(c_t), \quad w_s \text{ given}$$

$0 < \beta < 1$.

$$w_{t+1} = \tilde{r}_t (w_t - c_t)$$

$u' > 0, u'' < 0$.

IID random variables.

Constant discount factor β; u independent of t.

STATIONARY (infinite horizon) DYNAMIC PROGRAM.

$\liminf_{t \to \infty} \cancel{\beta^t} w_t$ exists and is ≥ 0.

$w_t \geq 0$ for all time. (*)

"OBVIOUSLY" (?),

$V_s(w) = V_{s'}(w)$ for all s, s'!

max value function. J_s.

$$V(w) = \max_c \{u(c) + \beta \mathbb{E}\, V(\tilde{r}(w - c))\}$$

$c(w) = \arg\max_c \{u(c) + \beta \mathbb{E}\, V(\tilde{r}(w - c))\}$

next period's uncertain wealth

V appears on both left and right hand sides.

Fixed point in function space.

General stationary problem

$$\max_{\underset{\sim}{u}} \mathbb{E}_s \sum_{t=s}^{\infty} \beta^{t-s} f(x_t, u_t)$$

states → x_t, controls → u_t

s.t. $u_t \in F(x_t)$ $(t = s, s+1, \ldots)$ $\quad (0 < \beta < 1)$

$P(x_{t+1} \mid x_t, u_t)$ — ind. of t.

$\boxed{\text{Transversality constraint.}}$

Bellman equation.

$$\left.\begin{array}{l} V(x) = \\ u^*(x) = \arg \end{array}\right\} \max_{u \in F(x)} \left\{ f(x,u) + \beta\, \mathbb{E}\left[V(\tilde{x}) \mid x, u\right] \right.$$

Find V, u^* to solve Bellman equation.

Will be sol^n if f is BOUNDED.

Otherwise:

Check that V is the value of choosing $u^*(x)$.

$$u(c) = \frac{c^{1-\varepsilon}}{1-\varepsilon} \quad (\varepsilon > 0, \varepsilon \neq 1)$$
$\ln c$ if $\varepsilon = 1$

$$V_s(w_s) = \frac{\alpha_s w_s^{1-\varepsilon}}{1-\varepsilon} \Big\} \text{ finite horizon.}$$

$$c_s(w_s) = \gamma_s w_s.$$

As $T \to \infty$,
$c_s^T(w_s) \to \gamma w_s$,
$V_s^T(w_s) \to \frac{\alpha w_s^{1-\varepsilon}}{1-\varepsilon}$

CONJECTURE. When $T = \infty$,

$$V(w) = \frac{\alpha w^{1-\varepsilon}}{1-\varepsilon}, \quad c(w) = \gamma w.$$

$$\frac{\alpha w^{1-\varepsilon}}{1-\varepsilon} = \Big\{ \max_c \frac{c^{1-\varepsilon}}{1-\varepsilon} + \beta \mathbb{E} \frac{\alpha [\tilde{r}(w-c)]^{1-\varepsilon}}{1-\varepsilon}.$$

$\gamma w = \arg$

$$\cancel{= \frac{w^{1-\varepsilon}}{1-\varepsilon}\left[\gamma^{1-\varepsilon} + \beta\alpha(1-\gamma)^{1-\varepsilon}\,\mathbb{E}\,\tilde{r}^{1-\varepsilon}\right]}$$

$$\cancel{= \frac{1}{1-\varepsilon}\left[c^{1-\varepsilon} + \beta\alpha[(1-\gamma)w]^{1-\varepsilon}\,\mathbb{E}\,\tilde{r}^{1-\varepsilon}\right]}$$

As $T \to \infty$,

$$\cancel{\frac{\alpha}{1-\varepsilon} = \Big\{ \max \gamma^{1-\varepsilon}}$$
$$\cancel{\gamma = \arg}$$

$$c_s^T(w_s) = \gamma_{T-s} w_s \to \gamma w_s \text{ as } T \to \infty.$$

$$V_s^T(w_s) = \frac{\alpha_{T-s} w_s^{1-\varepsilon}}{1-\varepsilon} \to \frac{\alpha w_s^{1-\varepsilon}}{1-\varepsilon}.$$

(12) $$\frac{\alpha}{1-\varepsilon} = \max_{\gamma} \frac{\gamma^{1-\varepsilon}}{1-\varepsilon} + \beta \frac{\mathbb{E}[\alpha(1-\gamma)\tilde{r}]^{1-\varepsilon}}{1-\varepsilon}$$

$$= \max_{\gamma} \frac{\gamma^{1-\varepsilon}}{1-\varepsilon} + \frac{\beta[\alpha(1-\gamma)]^{1-\varepsilon}}{1-\varepsilon} \mathbb{E}\tilde{r}^{1-\varepsilon}$$

FOC $\gamma^{-\varepsilon} = \beta\alpha^{1-\varepsilon}(1-\gamma)^{-\varepsilon}\,\mathbb{E}\tilde{r}^{1-\varepsilon}$.

Solve to find γ, as a function of $\alpha, \beta, \mathbb{E}\tilde{r}^{1-\varepsilon}$

Substitute to find α.

Candidate optimal policy.

$$\gamma = (\beta\alpha^{1-\varepsilon})^{-\frac{1}{\varepsilon}}(1-\gamma)(\mathbb{E}\tilde{r}^{1-\varepsilon})^{-1/\varepsilon}.$$

σ

substitute to find α.

13

Suppose that $f(x,u)$ is bounded both above and below. } BOUNDED CASE.

$M_1 \le f(x,u) \le M_2$ for all x, u.

Bellman eq.

$$V(x) = \max_{u \in F(x)} \left\{ f(x,u) + \beta \mathbb{E}\left[V(\tilde{x}) \mid x, u\right] \right\}$$

$u^*(x) = \arg$

$$0 < \beta < 1.$$

THEOREM

Under this boundedness condition, the Bellman eq. has a unique soln: $V(x)$ for each x, and any optimizing $u^*(x)$ is an optimal policy.

Sketch of Proof.

Given any function $V(x)$ satisfying

$$\frac{M_1}{1-\beta} \le V(x) \le \frac{M_2}{1-\beta} \text{ for all } x,$$

$$\sum_{t=s}^{\infty} \beta^{t-s} f(x_t, u_t) \in \left[\frac{M_1}{1-\beta}, \frac{M_2}{1-\beta}\right]$$

Define $[T^u V](x) = f(x,u) + \beta \mathbb{E}[V(\tilde{x}) \mid x, u]$

and $(T^* V)(x) = \max_{u \in F(x)} (T^u V)(x)$.

(14)

~~Prove that~~

$\cancel{(T^* V_1)}$

Define $d(V_1, V_2) = \sup_{x \in X} |V_1(x) - V_2(x)|$

[Finite because V_1, V_2 are bounded].

Prove that $d(T^* V_1, T^* V_2) \leq \beta d(V_1, V_2)$.

Contraction mapping.

Bellman has a unique solution

$$V = T^* V = T^{u^*} V$$

u^* - optimal policy.

* This will be an optimum

(needs checking?.

A1

EXAMPLES OF DANGER!!

PP. 461-2, Counterexamples.

Example 2. $0 < \beta < 1.$

$$\text{Max} \sum_{t=0}^{\infty} \beta^t (1 - u_t). \quad u_t \in [0,1]$$

$$\text{ST} \quad x_{t+1} = \frac{1}{\beta}(x_t + u_t), \quad x_0 > 0.$$

$$\left(\sum \frac{1}{\beta} x_t \to \infty. \right).$$

Objective $f(x,u) = 1 - u \in [0,1]$ is bounded.

Bellman equation:

$$\left.\begin{array}{l} J(x) = \\ u^*(x) = \arg \end{array}\right\} \max_u \left[1 - u + \beta J\left(\tfrac{1}{\beta}(x+u)\right) \right]$$

Look for solution with $J(x) = \gamma + x$.

$$\gamma + x = \max_u \; 1 - u + \beta \left[\frac{1}{\beta}(x+u) + \gamma \right]$$

$$= \max_u \; 1 + x + \beta\gamma$$

$u^*(x)$ arbitrary in $[0,1]$. $\quad \gamma = 1 + \beta\gamma, \; \gamma = \frac{1}{1-\beta}.$

[UNBOUNDED]

EVEN IF f is BOUNDED. (A2)

~~the~~ a solution to the Bellman equation could be inappropriate (if it is UNBOUNDED).

WHAT WE NEED:

A solution $J(x)$, $u^*(x)$.
to the Bellman equation
for which J is the value of
following the policy u^*.

In our example,

$$\max \sum_{t=0}^{\infty} \beta^t (1-u_t) \quad \text{ST } u_t \in [0,1].$$

Solution: choose $u^* = 0$ all t.

Then $J^{u^*}(x) = \sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta}$

This is an alternative sol^n.
to the Bellman equation: (BOUNDED).

BELLMAN

A3

$$\left.\begin{array}{l} J(x) = \\ u^*(x) = \arg \end{array}\right\} \max_u \left\{ 1 - u + \beta J\left(\frac{1}{\beta}(x+u)\right) \right\}$$

TRY $J(x) =$ constant (ind. of x).
(c)

$$\left.\begin{array}{l} c = \\ u^*(x) = \arg \end{array}\right\} \max \left\{ 1 - u + \beta \frac{c}{\cancel{\beta}} \right\}.$$

$u^* = 0$, $c = 1 + \cancel{\frac{\beta}{\beta}}\beta c$. $c = \frac{1}{1-\beta}$.

$\cancel{\beta c = \beta + c}$

If return function is bounded, there is a UNIQUE BOUNDED solution to the Bellman equation, which is the right one.

$[0 < \beta < 1]$

A6

EXAMPLES of economically interesting unbounded utility functions.

$$u(c) = \begin{cases} \dfrac{c^{1-\varepsilon}}{1-\varepsilon} \\ \ln c \quad (\text{if } \varepsilon = 1). \end{cases}$$

bounded below, not above iff $0 \leq \varepsilon < 1$.

bounded above, not below, iff $1 < \varepsilon \leq \infty$.

$\to$ unbounded both above and below.

$$u'(c) = c^{-\varepsilon}.$$

$$u(c) = -e^{-\alpha c} \quad (\alpha > 0)$$

bounded above; bounded below if we impose $c \geq 0$.

$$u(c) = -\frac{1}{2}(c - c^*)^2. \; (\cancel{c^* > 0}).$$

DITTO

~~iff~~ if $0 \leq c \leq c^*$.

~~RAMSEY~~

~~A3~~ A5.

LOWER BOUND CASE.

Suppose $f(x, u) \geq \gamma$
for all x, u.
(where $\gamma = 0$ if $\beta = 1$).

A solution to the Bellman equation which equals the true value of the corresponding optimal policy will be optimal.

UPPER BOUND CASE. A6

$$f(x, u) \leq \gamma \quad (\text{all } x, u).$$

1) Find a solution to the Bellman equation which is also bounded above (by $\frac{\gamma}{1-\beta}$).

(necessary condition for it to be the value of a feasible policy).

2) Check that either:

(a) It is the unique solution that is bounded above by $\frac{\gamma}{1-\beta}$.

OR (b) It is the limit as $T \to \infty$ of T-horizon values (and policies).

A7

P. 462, Example 3.

$$\text{Max} \sum_{t=0}^{\infty} \beta^t x_t (u_t - \alpha)$$

ST $u_t \in [0, \alpha] = U$.

$\alpha\beta = 1$. $x_{t+1} = x_t u_t$, $x_0 > 0$ given.

$0 < \beta \leq 1$, $\alpha > 0$.

Objective bounded above by 0, but unbounded below.

Bellman equation:

$$\left.\begin{array}{l} J(x) = \\ u^*(x) = \arg \end{array}\right\} \max_{u \in U} \{x(u-\alpha) + \beta J(xu)\}$$

Look for solution $J(x) = \gamma x$.

$$\gamma x = \max_{u \in U} \{x(u-\alpha) + \beta \gamma x u\}$$

$$= \max_{u \in U} \{x u (1 + \beta\gamma) - \alpha x\}.$$

$u^*(x) = \alpha$, (> 0).

$\gamma = \alpha\beta\gamma = \gamma$, Solution for all $\gamma \geq 0$.

A8

Check ~~that~~ J is the actual value of the policy.

Choose u = constant.

$$x_{t+1} = x_t u$$

$$\cancel{= x_0}$$

$$x_t = x_0 u^t.$$

$$\sum_{t=0}^{\infty} \beta^t x_t (u - \alpha)$$

$$= \sum_{t=0}^{\infty} x_0 (\beta u)^t (u - \alpha)$$

$$= \frac{x_0 (u-\alpha)}{1 - \beta u} \quad (\text{if } \beta u < 1),$$

or $u < \alpha$.

A9

PROCEDURES for solving Bellman equation:

1) Finite horizon, then take limits. (guess?)

2) Read KLJVDD -

3) POLICY IMPROVEMENT.

P.458, (17). § 12.7.

$$\max_u \boxed{E}\left[\sum_{t=0}^{\infty} \beta^t f(x,u)\right]$$

$$x_{t+1} = g(x_t, u_t, v_{t+1})$$

random shocks.

$P(v_{t+1} | v_t)$ specified.

$u \in U$, x_0 given.

$J(x,v)$, $u^*(x,v)$.

A10

$$J(x,v) = \max_{u\in U} \{ f(x,u) + \beta \mathbb{E}[J(x',v')|x,v] \}$$

$$x' = g(x,u,v'). \quad P(v'|v)$$

$$(T^u J)(x,v) = f(x,u) + \beta \mathbb{E}[J(x',v')|x,u,v]$$

$$(T^* J)(x,v) = \max_{u\in U} (T^u J)(x,v).$$

Bellman equation $J = T^* J = T^{u^*} J$

POLICY IMPROVEMENT.

Start with arbitrary $u_0(x,v)$. $(x \in X$

Find its value, which satisfies

$$J_0 = J^{u_0} = T^{u_0} J_0.$$

$$T^* J_0 = T^{u_1} J_0.$$

$$J_1 = J^{u_1} = T^{u_1} J_1.$$

$$T^* J_{k-1} = T^{u_k} J_{k-1}.$$

$$J_k = J^{u_k} = T^{u_k} J_k.$$