0

DIFFEREN$^{\text{CE}}_{\text{TIAL}}$ EQUATIONS.

What's the difference?

$l_0, r_0 \quad l_1 \quad r_2 \quad l_3 \quad r_3 \quad \ldots$

left and right feet.

time DISCRETE.

$h(t)$. – continuous fn. of time.

$$l_{2m+1} = \lambda(r_{2m})$$

$$r_{2m+2} = \rho(l_{2m+1})$$

↘ coupled equations.

DIFFERENCE equation:

$$x_{t+1} - x_t = \boxed{\Delta x_t = d_t(x_t)}.$$

Recurrence relation

$$x_{t+1} = r_t(x_t).$$

$x_0, x_1, \dots, x_T$.

as functions of x_0. $t = 0, \dots, T-1$

LINEAR CASE.

$$\boxed{x_{t+1} - x_t = d_t} \quad (t = 0, \dots, T-1).$$

T equations in $T+1$ unknowns.

$(x_0, x_1, \dots, x_T)$. Rank T (check). (upper trian

One degree of freedom.

BUT. if an initial condition is given (x_0

or a terminal condition (x_T).

then $x_t = \cancel{\sum_{s=0}^{t}} x_0 + \sum_{s=0}^{t-1} d_s$.

or $x_t = x_T -$ (exercise)...

2

Wealth equation (in discrete time).

$$w_{t+1} = (1+r_t)(w_t + y_t - e_t).$$

rate of interest. income expenditure

$$w_{t+1} = \rho_t (w_t - x_t)$$

gross return

net expenditure.

$$= \rho_t (w_t + s_t) \quad (*)$$

net saving.

Interest compounded:

$$R_t = \prod_{k=0}^{t-1}(1+r_t) = \prod_{k=0}^{t-1} \rho_t .$$

gross return factor

$(1+r)^t$ when $r_t = r$ (all t).

$$R_{t+1} = (1+r_t)R_t .$$

Discounted wealth:

$$\omega_t := w_t / R_t .$$

$$\omega_{t+1} = \frac{w_{t+1}}{R_{t+1}} = \frac{1+r_t}{R_{t+1}}(w_t - x_t) = \frac{1}{R_t}(w_t - x_t)$$

Present discounted value of future expenditure.

$\cancel{W_t := w_t}$

$$\xi_t := x_t / R_t .$$

$$\boxed{R_0 = 1.}$$

product of zero terms

$$\omega_{t+1} = \omega_t - \xi_t .$$

$$\boxed{\omega_t = \omega_0 - \sum_{k=0}^{t-1} \xi_k}$$

$$\cancel{w_t = R_t w_0 - \sum_{k=0}^{t-1} \prod_{j=k}^{t-1} R_j w_j .}$$

VECTOR Linear difference equation in n variables

$$x_{t+1} = A_t x_t + d_t.$$

n vector — n×n matrix. — n vector.

Sol^n: $$\boxed{x_t = P_{t,0}\, x_0 + \sum_{k=1}^{t} P_{t,k}\, d_{k-1}}$$

When $t=0$, $P_{0,0} = I$, $\cancel{P_{t,k}}$

~~When $t=1$, $P_{1,0} = I$, $P_{1,1} = A_1$~~

$x_1 = A_0 x_0 + d_0 \qquad P_{1,0} = A_0, \; P_{1,1} = I.$

$x_2 = A_1 x_1 + d_1 = A_1 A_0 x_0 + A_1 d_0 + d_1$

$P_{2,0} = A_1 A_0, \; P_{2,1} = A_1,$
$P_{2,2} = I.$

$$P_{t,0} = A_{t-1} A_{t-2} \cdots\cdots A_0$$

$$P_{t,k} = A_{t-1} A_{t-2} \cdots\cdots A_k$$

$$P_{t,t} = I.$$

4

Constant coefficients

$$x_{t+1} = Ax_t + d_t.$$

$$x_t = A^t x_0 + \sum_{k=1}^{t} A^{t-k} d_k$$

$$A^0 = I.$$

Autonomous

Constant right hand side

$$x_{t+1} = Ax_t + d \qquad (x_{t+1} = x_t \ldots$$

Steady state x^*, $x^* = Ax^* + d.$

$x^* = (I-A)^{-1} d$ (if $(I-A)^{-1}$ exi...

INSERT 4...

Condition for stability

$x_t \to x^*$ as $t \to \infty.$

~~Generally~~. Requires $A^t d \to 0$ as $t \to \infty.$

~~$A^t x_0 \to 0$ as $t \to \infty.$~~

$A^t \to 0$. as sufficient condition for all d.

Characteristic roots

$$|A - \lambda I| = 0.$$

Each eigenvalue has modulus < ...
(even if complex)

$|\lambda + i\mu| = \sqrt{\lambda^2 + \mu^2}$ — distance from origin in Argand diagram

4 bis

$$x_{t+1} = Ax_t + d$$

Sol^n: $$x_t = A^t x_0 + \sum_{k=1}^{t} A^{t-k} d$$

$$= A^t x_0 + S_t d$$

where $$S_t = I + A + \dots + A^{t-1}$$

$$= \sum_{k=1}^{t} A^{t-k}$$ — geometric series in the square matrix A

$$1 + a + a^2 + \dots + a^{t-1} = \frac{1-a^t}{1-a}$$

(multiply ~~equ~~ each side by $1-a$).

$$(I-A) S_t = I + A - A + A^2 - A^2 \dots - A^t$$

$$= I - A^t$$

$$S_t = (I-A)^{-1}(I-A^t).$$ assuming $(I-A)^{-1}$ exists.

$$x_t = A^t x_0 + (I-A)^{-1}(I-A^t) d.$$

PROVIDED $(I-A)^{-1}$ exists.

$$x_t \to (I-A)^{-1} d = x^*$$ as $t \to \infty$. $\boxed{\text{provided } A^t \to 0.}$

4 ter.

EXAMPLE.

$$A^t d \to 0$$

$$\not\Rightarrow A^t \to 0$$

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}, \quad d = (1, 0)$$

$$A^t = \begin{pmatrix} 2^{-t} & 0 \\ 0 & 2^t \end{pmatrix} \not\to \begin{pmatrix} 0, & 0 \\ 0, & 0 \end{pmatrix}$$

as $t \to \infty$

$$A^t d = \begin{pmatrix} 2^{-t} & 0 \\ 0 & 2^t \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2^{-t} \\ 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ as } t \to \infty.$$

BUT $A^t d = 0$ for all $d \Rightarrow A^t \to 0$.

$A^t d \to 0$ for $d = e_j$ (jth unit vector) $\Rightarrow$ jth col. of $A^t \to 0$

SUPPOSE A has n distinct eigenvalues (real or complex)

A may not be symmetric.

Then the eigenvectors form a (complex) matrix P such that

$$P^* A P = \mathrm{diag}(\underbrace{\lambda_1, \dots, \lambda_n}_{\text{eigenvalues (real or complex)}}).$$

adjoint matrix

P^*_{jk} is the complex conjugate of P_{ji}.

$$p_{jk} + i\, q_{jk} = p_{kj} - i\, q_{kj}$$

Take ~~conjugate complex~~ complex conjugate of transpose.

$P^* P = I$. (Spectral Theorem)

6

$$x_{t+1} = A x_t + d$$

~~Define $y_t = P x_t$, ?~~

~~$P^* x_{t+1} =$~~

Define $x_t = P y_t$.

$$y_t = P^* x_t$$

$$y_{t+1} = P^* x_{t+1} = P^* A P y_t + P^* d$$

$$\Rightarrow y_{t+1} = \mathrm{diag}(\lambda_1, \dots, \lambda_n) y_t + P^* d$$

UNCOUPLED equations

$$y^j_{t+1} = \lambda_j y^j_t + (P^* d)_j.$$

n separated equations

STABILITY iff $\lambda_j^t \to 0$ as $t \to \infty$ (all j)

$$\Leftrightarrow |\lambda_j| < 1.$$

A diagonalizable matrix is stable ($A^t \to 0$) iff its eigenvalues are all < 1 in absolute value (modulus)

Also for non-diagonalizable matrices.

Second-order equation:
(linear, constant coefficients, homogeneous).

$$x_{t+1} + a x_t + b x_{t-1} = 0$$

Define $y_t = x_{t-1}$

Then $\left. \begin{aligned} x_{t+1} &= -a x_t - b y_t \\ y_{t+1} &= x_t \end{aligned} \right\}$

Coupled pair of 1st order equations

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

Instead of treating coupled pair, consider 2nd equation directly.

Guess ~~when~~ that $x_t = \lambda^t x_0$ could be a solution, for suitable λ, x_0.

$$\lambda^{t+1} x_0 + a \lambda^t x_0 + b \lambda^{t-1} x_0 = 0.$$

Cancel $\lambda^{t-1} x_0$ ($\neq 0$, assume).

$$\lambda^2 + a\lambda + b = 0.$$

Roots λ_1, λ_2. ($\lambda_1 \neq \lambda_2$). Case of distinct roots

~~Bas~~ Possible Solutions:

$x_t = \lambda_1^t, \quad x_t = \lambda_2^t.$ if $x_0 = 1$.

Two degrees of freedom.

General solⁿ: $x_t = \alpha\lambda_1^t + \beta\lambda_2^t.$

COINCIDENT ROOTS.

If $\lambda_1 = \lambda_2 = \lambda$, solⁿs $x_t = \lambda^t, \quad t\lambda^t.$

General solⁿ: $x_t = (\alpha + \beta t)\lambda^t$

Two degrees of freedom.

~~If~~ (to HOMOGENEOUS equation)

$x_{t+1} + a x_t + b x_{t-1} = 0$

INHOM. eq., general RHS, d_t.

Find a particular solution x_t^p.

General solⁿ: $x_t = x_t^p$ + solⁿ to homog. equation.

two degrees of freedom.

AUTONOMOUS.

LOCAL STABILITY of non-linear system of difference equations:

$$x_{t+1} = F(x_t). \quad F: \mathbb{R}^n \to \mathbb{R}^n$$

stationary state, if it exists, must be a fixed point of F.

$$x^* = F(x^*)$$

$$x_{t+1} - x^* = F(x_t) - F(x^*)$$

$$\simeq F'(x^*)(x_t - x^*) \qquad (\text{if } x_t \to x^*).$$

$F'(x^*)$: $(n \times n)$ - matrix of derivatives $\frac{\partial F_i}{\partial x_j}$.

JACOBIAN matrix.

LOCAL Stability requires $F'(x^*)$ to be a stable matrix. (all eigenvalues have modulus or absolute value < 1).

A1

$$\underset{\sim}{x}_{t+1} = \underset{\sim}{A}\,\underset{\sim}{x}_t,$$

$$\underset{\sim}{x}_{t+1} - \underset{\sim}{A}\,\underset{\sim}{x}_t = \underset{\sim}{0}$$ homogeneous equation in n variables

constant. $n \times n$. matrix

General solution: (?)

$$\underset{\sim}{x}_t = \sum_{j=1}^{n} \underset{\sim}{c}^j \lambda_j^t$$

sum of n linearly independent power solutions. $\underset{\sim}{c}^j \lambda_j^t$

When are these n solutions linearly independent?

Answer: iff $\sum_{j=1}^{n} \underset{\sim}{c}^j \lambda_j^t \equiv 0 \Rightarrow$ each $\underset{\sim}{c}^j = \underset{\sim}{0}$

$\Longleftrightarrow$ all λ_j different (n distinct roots).

A2

When is $\sum_{j=1}^{n} \underset{\sim}{c}^{j} \lambda_j^t$ a solution?? (of $x_{t+1} = A x_t$). vector eq.n.

~~Answer: iff~~

$$\cancel{\sum_{j=1}^{n} \underset{\sim}{c}^{j} \lambda_j^{t+1} - A \sum_{j=1}^{n} \underset{\sim}{c}^{j} \lambda_j^{t} \equiv 0}$$

$$\cancel{\text{or } \sum_{j=1}^{n} \underset{\sim}{c}^{j} (\lambda_j^{t+1}}$$

When is $\underset{\sim}{c}^{j} \lambda_j^t$ a solution?

Answer: iff $\underset{\sim}{c}^{j} \lambda_j^{t+1} - \underset{\sim}{A}\, \underset{\sim}{c}^{j} \lambda_j^{t} \equiv 0$

~~or (assuming~~ $\cancel{c^j = \lambda}$.

or iff $\lambda_j^t (\underset{\sim}{c}^{j} \lambda_j - \underset{\sim}{A}\, \underset{\sim}{c}^{j}) = 0$ (all t).

iff $\lambda_j = 0$ or $\boxed{\underset{\sim}{A}\, \underset{\sim}{c}^{j} = \lambda_j \underset{\sim}{c}^{j}}$

either $\underset{\sim}{c}^{j} = \underset{\sim}{0}$ or λ_j an eigenvalue and $\underset{\sim}{c}^{j}$ and eigenvector.

Providing the eigenvalues λ_j $(j=1, \dots, n)$ are distinct (possibly one $\lambda_j = 0$), the general solution is $\underset{\sim}{x}_t = \sum_{j=1}^{n} \underset{\sim}{c}^{j} \lambda_j^t$, where $\underset{\sim}{c}^{j}$'s are eigenvectors.

A3

What if eigenvalues are not distinct?

<u>Example</u> $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}$$

$$= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$

$$= |A| - \lambda \underbrace{\operatorname{tr} A}_{a_{11}+a_{22}} + \lambda^2$$

$$= (\lambda - \lambda_1)^2 = \lambda^2 - 2\lambda\lambda_1 + \cancel{2}\lambda_1^2$$

$$\lambda_1 = \tfrac{1}{2} \operatorname{tr} A = \tfrac{1}{2}(a_{11} + a_{22})$$

$$\lambda_1^2 = \det A.$$

$$x_{t+1} = a_{11} x_t + a_{12} y_t$$

$$y_{t+1} = a_{21} x_t + a_{22} y_t.$$

Make $a_{12} = a_{21} = 0$ (diagonal).

$$a_{11} = a_{22} = a.$$

$$x_{t+1} = a x_t,$$

$$y_{t+1} = a y_t.$$

A6

~~$x_{t+1} - 2\lambda x_t +$~~

$$x_{t+1} + a x_t + b x_{t-1} = 0.$$

$$x_t = c\lambda^t,$$

$$c\lambda^{t+1} + ac\lambda^t + bc\lambda^{t-1} = 0$$

If $c \neq 0, \lambda \neq 0$, $\lambda^2 + a\lambda + b = 0$
$(\div c\lambda^{t-1})$.

Two roots λ_1, λ_2.
All well if $\lambda_1 \neq \lambda_2$

general sol^n: $x_t = c_1\lambda_1^t + c_2\lambda_2^t$.

BUT, what if $\lambda_1 = \lambda_2$? $(= \lambda^*)$.

$$x_{t+1} - 2\lambda^* x_t + \lambda^{*2} x_{t-1} = 0.$$

$$(\lambda^2 - 2\lambda^*\lambda + \lambda^{*2} = 0)$$

$$(\lambda - \lambda^*)^2 = 0.$$

Put $y_t = \lambda^{*-t} x_t$ $\quad (x_t = c\lambda^{*t})$ one sol^n

~~$x_{t+1} + a$~~

$$(\lambda^*)^{-t-1} y_{t+1} - 2\lambda^* (\lambda^*)^{-t} y_t + (\lambda^{*2})(\lambda^*)^{-t+1} y_{t-1} = 0$$

If $\lambda^* \neq 0$, $y_{t+1} - 2y_t + y_{t-1} = 0$.

A5

Put $z_t = y_{t+1} - y_t$.

Then $z_t - z_{t-1} = 0$.

So

$$z_t = c, \quad y_t = \alpha + \beta t$$

$$x_t = y_t \lambda^{*t} = (\alpha + \beta t)\lambda^{*t}.$$

λ

lin. in t.

Solution with a double eigenvalue.

With a triple eigenvalue.

$$x_t = (\alpha + \beta t + \gamma t^2)\lambda^{*t}.$$

COMPLEX EIGENVALUES.

$$\cancel{x_t = (\alpha + i\beta)(\lambda + i\mu)^t + (\gamma + i\delta)(\lambda - i\mu)^t}$$

~~complex~~ conjugates. eigenvalues

$$\lambda_1, \lambda_2 = \lambda \pm i\mu.$$

$$x_t = (\alpha + i\beta)(\lambda + i\mu)^t + (\gamma + i\delta)(\lambda - i\mu)^t$$

$$\lambda + i\mu = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$\lambda - i\mu = re^{-i\theta} = r(\cos\theta - i\sin\theta)$$

DE MOIVRE.

A6

$$x_t = (\alpha + i\beta) r^t e^{it\theta} + (\gamma + i\delta) r^t e^{-it\theta}$$

$$= (\alpha + i\beta) r^t (\cos t\theta + i \sin t\theta)$$
$$+ (\gamma + i\delta) r^t (\cos t\theta - i \sin t\theta).$$

$$= (\alpha + \gamma) r^t \cos t\theta - (\beta - \delta) r^t \sin t\theta$$
$$i r^t \cancel{\beta \cos t\theta} [(\beta + \delta) \cos t\theta + (\alpha - \gamma) \sin t\theta]$$
$$\cancel{+ i\alpha} + \text{complex terms} \; (=0$$

$$\beta + \delta = 0, \quad \alpha - \gamma = 0$$

$$= 2\alpha r^t \cos t\theta - 2\beta r^t \sin t\theta.$$

$\to 0$ iff $|r| < 1$.

$\to \infty$ iff $|r| > 1$

circles iff $|r| = 1$.