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Lecture 2: Real Analysis,
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1 Finite-Dimensional Euclidean Space

1.1 Preliminaries

Let $\mathbb{N} := \{1, 2, \ldots\}$ denote the countably infinite set of *natural numbers*. For any natural number $K \in \mathbb{N}$, the K-dimensional real (Euclidean) space is the K-fold Cartesian product of $\mathbb{R}$. We denote this space by $\mathbb{R}^K$, so that $x \in \mathbb{R}^K$ is $(x_1, x_2, \ldots, x_K) = (x_i)_{i=1}^K$. Sometimes we abuse notation by letting K denote the set $\{1, 2, \ldots, K\}$. Then the typical member of $\mathbb{R}^K$ can be denoted by $(x_i)_{i \in K}$.

Remark 1. *The textbooks EMEA and FMEA typically use* vector notation*, where the typical member of* $\mathbb{R}^K$ *is treated as a* vector *written as* $\mathbf{x}$ *instead of* x*. This illustrates the proposition that neither mathematicians nor economists use consistent notation!*

The *origin* of $\mathbb{R}^K$ is the vector 0 whose components $(0, 0, \ldots, 0)$ are all zero.

Given any pair $a, b \in \mathbb{R}$ there are four different possible inequalities, namely: $a > b$, $a \geq b$, $a \leq b$ and $a < b$. If $a \neq b$, exactly one of these holds. But if $a = b$, then both $a \geq b$ and $a \leq b$.

Given any pair $x, y \in \mathbb{R}^K$ where $\#K \geq 2$, there are six different possible inequalities, namely: $x \gg y$, $x > y$, $x \geqq y$, $x \leqq y$, $x < y$, and $x \ll y$. The first three inequalities are defined so that:

1. $x \gg y$ iff $x_i > y_i$ for all $i \in K$;
2. $x > y$ iff $x \neq y$ and $x_i \geq y_i$ for all $i \in K$;
3. $x \geqq y$ iff $x_i \geq y_i$ for all $i \in K$.

Clearly $x \leqq y$ iff $y \geqq x$, etc. Given any pair $x, y \in \mathbb{R}^1$, of course, one has $x \gg y$ iff $x > y$. But in $\mathbb{R}^K$ when $\#K \geq 2$, none of the six inequalities may hold, as happens when $x = (1, 0)$ and $y = (0, 1)$ in $\mathbb{R}^2$.

Yet more notation: the *non-negative orthant* in $\mathbb{R}^K$ is the set $\mathbb{R}^K_+ := \{x \in \mathbb{R}^K \mid x \geqq 0\}$, whereas the *positive orthant* is $\mathbb{R}^K_{++} := \{x \in \mathbb{R}^K \mid x \gg 0\}$. When $\#K = 2$ these are *quadrants*, and when $\#K = 3$ these are *octants*. There is no special notation for the set $\mathbb{R}^K_+ \setminus \{0\} = \{x \in \mathbb{R}^K \mid x > 0\}$.

1.2 The Euclidean Metric

In R^1 the *distance* $d(a, b)$ between a and b is $\|a - b\|$. In R^2, Pythagoras' theorem implies that the *distance* $d(x, y)$ between $x = (x_1, x_2)$ and $y = (y_1, y_2)$ is the positive solution to the equation $d^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2$. Even in R^3, the *distance* $d(x, y)$ between $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is the positive solution to the equation $d^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2$. All of these three cases are covered by the single formula $d(x, y) = \sqrt{\left(\sum_{i \in K}(x_i - y_i)^2\right)}$.

Accordingly, for general $K \in \mathbb{N}$ we have:

DEFINITION 1. *The* Euclidean distance $d(x, y)$ *between any two points* $x, y \in \mathbb{R}^K$ *is*

$$d(x, y) = \left(\sum_{i \in K} (x_i - y_i)^2 \right)^{1/2} .$$

The space $X = \mathbb{R}^K$ equipped with its Euclidean distance $d : X \times X \to \mathbb{R}$ is a prominent example meeting the next definition.

DEFINITION 2. *Given any set* X*, the function* $d : X \times X \to \mathbb{R}$ *with* $(x, y) \mapsto d(x, y)$ *is a* metric *on* X*, and the pair* (X, d) *is called a* metric space*, provided that:*

- $d(x, y) \geq 0$ *for all* $x, y \in X$*, with* $d(x, y) = 0$ *iff* $x = y$*;*
- *the* triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$ *holds for all* $x, y, z \in X$*.*

1.3 The Euclidean Norm

The finite-dimensional Euclidean space $\mathbb{R}^K$ is a prominent example of:

DEFINITION 3. *A* real linear (or vector) space *is a set* L *equipped with the two binary operations* $(x, y) \mapsto x + y$ *from* $L \times L$ *to* L *and* $(\lambda, x) \mapsto \lambda x$ *from* $\mathbb{R} \times L$ *to* L*, as well as a unique* zero vector $0 \in L$*, which for all* $x, y, z \in L$ *and all* $\lambda, \mu \in \mathbb{R}$ *must satisfy:*

1. $x + y = y + x$ *(addition commutes);*
2. $(x + y) + z = x + (y + z)$ *(addition is associative, so* $x + y + z$ *is well defined, as is any finite sum);*
3. $x + 0 = x$*;*
4. *for each* $x \in L$*, there is a unique* inverse $-x$ *such that* $x + (-x) = 0$*;*
5. $\lambda(\mu x) = (\lambda\mu)x$ *(scalar multiplication is associative, so* $\lambda\mu x$ *is well defined);*
6. $0x = 0$*;*
7. $1x = x$*;*
8. $(\lambda + \mu)x = \lambda x + \mu x$ *(first distributive law);*
9. $\lambda(x + y) = \lambda x + \lambda y$ *(second distributive law).*

In order to measure how far from 0 an element x of $\mathbb{R}^K$ is, we use the *Euclidean norm* which is defined as[1]

$$\|x\| = \left(\sum_{k=1}^{K} x_k^2 \right)^{1/2} .$$

It is obvious that when $K = 1$ the Euclidean norm corresponds to the absolute value. More importantly, it is also clear that for every $x \in \mathbb{R}^K$, one has: (i) $\|x\| \geq 0$; (ii) $\|x\| = 0 \Leftrightarrow x = 0$; (iii) $-y \leqq x \leqq y \Rightarrow \|x\| \leq \|y\|$; and (iv) $\|x\| = \|-x\|$. A final crucial property is the Triangle Inequality: given $x, y \in \mathbb{R}^K$, one has $\|x + y\| \leq \|x\| + \|y\|$.

[1] To avoid confusion, you can be explicit about the dimension for which the norm is being used, by adopting the notation $\|\cdot\|_K$ instead. Also, we will simplify the notation by not always writing the limits in the index of a summation, when it is obvious what these limits are; for instance, we may write $\left(\sum_k x_k^2\right)^{1/2}$ for the definition that follows.

2 Functions

Let X and Y be two nonempty sets. A *function* f from a set X into a set Y, denoted $f : X \to Y$, is a rule that assigns to each $x \in X$ a unique $f(x) \in Y$; here, set X is said to be the *domain* of f, and Y its *target set* or *co-domain.* If $f : X \to Y$ and $A \subseteq X$, the *image of* A *under* f, denoted by $f[A]$, is the set

$$f[A] = \{y \in Y | \exists x \in A : f(x) = y\}.$$

In particular, the image $f[X]$ of the whole domain is called the *range* of f.

Function $f : X \to Y$ is said to be *onto*, or *surjective*, if $f[X] = Y$; it is said to be *one-to-one*, or *injective*, if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

A function is said to be *a one-to-one correspondence*, or *bijective*, if it is both onto and one-to-one.

If $f : X \to Y$, and $B \subseteq Y$, the *inverse image of* B *under* f, denoted $f^{-1}[B]$, is the set

$$f^{-1}[B] = \{x \in X | f(x) \in B\}.$$

If $f : X \to Y$ is a one-to-one correspondence, the *inverse function* $f^{-1} : Y \to X$ is implicitly defined by $\{f^{-1}(y)\} = f^{-1}[\{y\}]$. Notice that this would not have been be a *bona fide* definition, had we forgotten to say that f is a one-to-one correspondence (what could have gone wrong?). The proof of the following theorem is left as an exercise.

THEOREM 1. *The function* $f : X \to Y$ *is onto iff for all non-empty* $B \subseteq Y$ *one has* $f^{-1}[B] \neq \varnothing$.

From now on, we only concentrate on definitions and concepts in Euclidean spaces and maintain the assumption that $K \in \mathbb{N}$.

3 Sequences

A *sequence* in $\mathbb{R}^K$ is a function $f : \mathbb{N} \to \mathbb{R}^K$. If no confusion is likely, the space in which a sequence lies is omitted. Following usual notation in mathematics, we can express sequences as $(a_1, a_2, \ldots)$ or $(a_n)_{n=1}^{\infty}$, where $a_n = f(n)$, for $n \in \mathbb{N}$.

EXAMPLE 1. *Suppose that* $f(n) = (\sqrt{n}, 1/n, 3) \in \mathbb{R}^3$, *for all* $n \in \mathbb{N}$. *Then we can express the sequence as* $((1, 1, 3), (\sqrt{2}, 1/2, 3), (\sqrt{3}, 1/3, 3), \ldots)$ *or* $(\sqrt{n}, 1/n, 3)_{n=1}^{\infty}$.

It is very important to notice that a sequence has more structure than a set (i.e., it is more complicated). Remember that a set is completely defined by its elements, no matter how they are described. For example, the set $\{0, 3, 8, 15, 24\}$ is the same as the set $\{24, 15, 8, 3, 0\}$. However, the sequences $(0, 3, 8, 15, 24, \ldots)$ and $(24, 15, 8, 3, 0, \ldots)$ are clearly different: in a sequence, the order matters!

Using the structure that a sequence has, $(a_n)_{n=1}^{\infty}$ is *nondecreasing* if for all $n \in \mathbb{N}$, $a_{n+1} \geq a_n$ and *nonincreasing* if for all n, $a_{n+1} \leq a_n$. If all the inequalities in the first definition are strict, the sequence is *increasing.* while if all the inequalities in the second definition are strict, the sequence is *decreasing.*

The sequence $(a_n)_{n=1}^{\infty}$ is *bounded above* if there exists $\bar{a} \in \mathbb{R}$ such that $a_n \leq \bar{a}$ for all n. It is *bounded below* if there exists $\bar{a} \in \mathbb{R}$ such that $a_n \geq \bar{a}$ for all n, and it is *bounded* if it is bounded both above and below.

Given a sequence $(a_n)_{n=1}^{\infty}$, a sequence $(b_m)_{m=1}^{\infty}$ is *a subsequence of* $(a_n)_{n=1}^{\infty}$ if there exists an increasing sequence $(n_m)_{m=1}^{\infty}$ such that $n_m \in \mathbb{N}$ and $b_m = a_{n_m}$ for all $m \in \mathbb{N}$. That is, a subsequence is a selection of some (possibly all) members of the original sequence that preserves the original order.

EXAMPLE 2. *Consider the sequence* $(1/\sqrt{n})_{n=1}^{\infty}$, *and note that* $(1/\sqrt{2n+5})_{n=1}^{\infty}$ *is a subsequence of the former. To see why, consider the sequence* $(n_m)_{m=1}^{\infty} = (2m+5)_{m=1}^{\infty}$.

EXERCISE 1. *Is* $(1/\sqrt{n})_{n=1}^{\infty}$ *a subsequence of* $(1/n)_{n=1}^{\infty}$? *How about the other way around?*

4 Limits

4.1 Limits of sequences

The point $a \in \mathbb{R}^K$ is a *limit* of the sequence $(a_n)_{n=1}^{\infty}$ if for all $\varepsilon > 0$ there exists some $n^* \in \mathbb{N}$ such that $\|a_n - a\| < \varepsilon$ for all $n \geq n^*$.

EXERCISE 2. *Does the sequence* $(1/\sqrt{n})_{n=1}^{\infty}$ *have a limit? Is it Cauchy? How about* $(\frac{3n}{n+\sqrt{n}})_{n=1}^{\infty}$?

Sequence $(a_n)_{n=1}^{\infty}$ is said to be *convergent* when it has a limit $a \in \mathbb{R}^K$. When $(a_n)_{n=1}^{\infty}$ converges to a, the following notation is also sometimes used: $a_n \to a$, or $\lim_{n\to\infty} a_n = a$.

EXERCISE 3. *Does* $((-1)^n)_{n=1}^{\infty}$ *converge? Does* $(-1/n)_{n=1}^{\infty}$?

It is convenient to allow $+\infty$ and $-\infty$ to be limits of sequences. Thus, we extend the definition as follows: for a sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$, we say that $\lim_{n\to\infty} a_n = \infty$ when for all $\Delta > 0$ there exists some $n^* \in \mathbb{N}$ for which one has that $a_n > \Delta$ for all $n \geq n^*$; we also say that $\lim_{n\to\infty} a_n = -\infty$ when $\lim_{n\to\infty}(-a_n) = \infty$.

EXERCISE 4. *Does the sequence* $(\frac{3n}{\sqrt{n}})_{n=1}^{\infty}$ *have a limit? Does it converge?*

The importance of concept introduced in the previous section is given by the following theorems, whose proofs are left as exercises.

THEOREM 2. *Sequence* $(a_n)_{n=1}^{\infty}$ *converges to* $a \in \mathbb{R}^K$ *if, and only if, every subsequence of* $(a_n)_{n=1}^{\infty}$ *converges to* a.

THEOREM 3. *If sequence* $(a_n)_{n=1}^{\infty}$ *in* $\mathbb{R}$ *is convergent, then it is bounded.*

THEOREM 4. *If a sequence* $(a_n)_{n=1}^{\infty}$ *is monotone and bounded, then it is convergent.*

THEOREM 5 (Bolzano-Weierstrass). *If sequence* $(a_n)_{n=1}^{\infty}$ *is bounded, then it has a convergent subsequence.*

An informal argument for the Bolzano-Weierstrass Theorem for sequences defined into $\mathbb{R}$ is as follows: if $(a_n)_{n=1}^{\infty}$ is bounded, then it lies in some bounded interval I_1. Slice that interval in halves. At least one of the halves will contain infinitely many terms of the sequence. Call that interval I_2, slice it in halves, and let I_3 be a half that contains infinitely many elements... By doing this indefinitely, we construct intervals $I_1, I_2, ...$ such that each I_n contains infinitely many terms of the sequence and $I_{n+1} \subseteq I_n$. By construction, we can find a subsequence $(x_{n_m})_{m=1}^{\infty}$ such that for all $m \in \mathbb{N}$, $a_{n_m} \in I_m$. This subsequence will have the property that their elements get arbitrarily close to one another as we move along the sequence (because, by construction, our "sequence" of intervals is in fact shrinking to zero diameter as m goes to ∞). Sequences with this property are said to be *Cauchy* and, in Euclidean spaces, are guaranteed to converge.

4.2 Limits of functions

Let $x \in \mathbb{R}^K$ and $\delta > 0$. The *open ball of radius δ around x*, denoted $B_\delta(x)$, is the set

$$B_\delta(x) = \{y \in \mathbb{R} | \|y - x\| < \delta\}$$

The *punctured open ball of radius δ around x*, denoted $B'_\delta(x)$, is the set $B'_\delta(x) = B_\delta(x) \setminus \{x\}$. A point $\bar{x} \in \mathbb{R}^K$ is *a limit point of $X \subseteq \mathbb{R}^K$* if for all $\varepsilon > 0$, $B'_\varepsilon(\bar{x}) \cap X \neq \varnothing$

Exercise 5. *Prove the following: Let $X \subseteq \mathbb{R}$. A point $\bar{x} \in \mathbb{R}$ is a limit point of X iff there exists a sequence $(x_n)_{n=1}^\infty$ defined in $X \setminus \{\bar{x}\}$ that converges to $\bar{x}$.*

Another type of limit has to do with functions, although not directly with sequences.

Definition. *Consider a function $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^K$. Suppose that $\bar{x} \in \mathbb{R}^K$ is a limit point of X and that $\bar{y} \in \mathbb{R}$. We say that $\lim_{x \to \bar{x}} f(x) = \bar{y}$ when for all $\varepsilon > 0$ there exists $\delta > 0$ for which one has that $|f(x) - \bar{y}| < \varepsilon$ for all $x \in B'_\delta(\bar{x}) \cap X$.*

It is important to notice that we do not require $\bar{x} \in X$ in our previous definition, so that $f(\bar{x})$ need not be defined. Also, one should notice that even if $\bar{x} \in X$, $\bar{x}$ is not always a limit point of X, in which case the definition does not apply. Finally, notice that even if $\bar{x} \in X$ and $\bar{x}$ is a limit point of X, it need not be the case that $\lim_{x \to \bar{x}} f(x) = f(\bar{x})$.

Definition. *Consider a function $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^K$. Suppose that $\bar{x} \in \mathbb{R}^K$ is a limit point of X. We say that $\lim_{x \to \bar{x}} f(x) = \infty$ when for all $\Delta > 0$, there exists $\delta > 0$ for which one has that $f(x) \geq \Delta$ for all $x \in B'_\delta(\bar{x}) \cap X$. We say that $\lim_{x \to \bar{x}} f(x) = -\infty$ when $\lim_{x \to \bar{x}} (-f)(x) = \infty$.*

Exercise 6. *Suppose that $X = \mathbb{R}$ and $(\forall x \in \mathbb{R})$ $f(x) = x + a$, for some $a \in \mathbb{R}$. What is $\lim_{x \to 0} f(x)$?*

Exercise 7. *Suppose that $X = \mathbb{R}$ and $f : X \to \mathbb{R}$ is defined by*

$$f(x) = \begin{cases} 1/x, & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

What is $\lim_{x \to 5} f(x)$? What is $\lim_{x \to 0} f(x)$?

Example 3. *Let $X = \mathbb{R} \setminus \{0\}$ and $f : X \to \mathbb{R}$ is defined by*

$$f(x) = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{otherwise.} \end{cases}$$

In this case, we claim that $\lim_{x \to 0} f(x)$ does not exist. To see why, fix $0 < \varepsilon < 1$, and notice that for all $\delta > 0$, there are $x_1, x_2 \in B_\delta(0)$ such that $f(x_1) = 1$ and $f(x_2) = -1$, and, hence, $|f(x_1) - f(x_2)| = 2 > 2\varepsilon$. Because of triangle inequality, it is thus impossible that for some $\bar{y} \in \mathbb{R}$, we have $|f(x_1) - \bar{y}| < \varepsilon$ and $|f(x_2) - \bar{y}| < \varepsilon$. Also, it is obvious that $\lim_{x \to 0} f(x) = \infty$ and $\lim_{x \to 0} f(x) = -\infty$ are both impossible.

There exists a tight relationship between limits of functions and limits of sequences, which is explored in the following theorem.

Theorem 6. *Consider a function $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^K$. Suppose that $\bar{x} \in \mathbb{R}^K$ is a limit point of X and that $\bar{y} \in \mathbb{R}$. Then, $\lim_{x \to \bar{x}} f(x) = \bar{y}$ if, and only if, for every sequence $(x_n)_{n=1}^\infty$ such that $x_n \in X \setminus \{\bar{x}\}$, for all $n \in \mathbb{N}$, and that $\lim_{n \to \infty} x_n = \bar{x}$, one has that $\lim_{n \to \infty} f(x_n) = \bar{y}$.*

Proof: We argue sufficiency by contradiction: suppose that for every sequence $(x_n)_{n=1}^{\infty}$ such that all $x_n \in X \setminus \{\bar{x}\}$ and that $\lim_{n\to\infty} x_n = \bar{x}$, we have that $\lim_{n\to\infty} f(x_n) = \bar{y}$, but, still, it is *not* true that $\lim_{x\to\bar{x}} f(x) = \bar{y}$. Then, there must exist some $\varepsilon > 0$ such that, for all $\delta > 0$,

$$\exists x \in B'_\delta(\bar{x}) \cap X : |f(x) - \bar{y}| \geq \varepsilon.$$

By assumption, for all $n \in \mathbb{N}$, there is $x_n \in B'_{1/n}(\bar{x}) \cap X$ for which $|f(x_n) - \bar{y}| \geq \varepsilon$. Construct the sequence $(x_n)_{n=1}^{\infty}$. By construction, $x_n^* \in X \setminus \{\bar{x}\}$ for all $n \in \mathbb{N}$, and, since $1/n \to 0$, we have that $\lim_{n\to\infty} x_n = \bar{x}$. However, by assumption, it is not true that $\lim_{n\to\infty} f(x_n) = \bar{y}$, which contradicts the initial hypothesis.

For necessity, consider any sequence $(x_n)_{n=1}^{\infty}$ such that all $x_n \in X\setminus\{\bar{x}\}$ and that $\lim_{n\to\infty} x_n = \bar{x}$. Fix $\varepsilon > 0$. Since $\lim_{x\to\bar{x}} f(x) = \bar{y} \in \mathbb{R}$, then, there is some $\delta > 0$ such that for all $x \in B'_\delta(\bar{x}) \cap X$ it is true that $|f(x) - \bar{y}| < \varepsilon$. Since $\lim_{n\to\infty} x_n = \bar{x}$, there is $n^* \in \mathbb{N}$ for which $x_n \in B_\delta(\bar{x})$ for all $n \geq n^*$. Moreover, since each $x_n \in X \setminus \{\bar{x}\}$, we have that, when $n \geq n^*$, $x_n \in B'_\delta(\bar{x}) \cap X$ and, therefore, $|f(x_n) - \bar{y}| < \varepsilon$. Since $\varepsilon > 0$ was arbitrarily chosen, this implies that $\lim_{n\to\infty} f(x_n) = \bar{y}$. *Q.E.D.*

4.3 Properties of limits

THEOREM 7. *Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$. Let $\bar{x}$ be a limit point of X. Suppose that for number $\bar{y}_1, \bar{y}_2 \in \mathbb{R}$ one has that $\lim_{x\to\bar{x}} f(x) = \bar{y}_1$ and $\lim_{x\to\bar{x}} g(x) = \bar{y}_2$. Then,*[2]

1. $\lim_{x\to\bar{x}}(f + g)(x) = \bar{y}_1 + \bar{y}_2$;
2. $\lim_{x\to\bar{x}}(\alpha f)(x) = \alpha\bar{y}_1$, *for all* $\alpha \in \mathbb{R}$;
3. $\lim_{x\to\bar{x}}(f.g)(x) = \bar{y}_1.\bar{y}_2$; *and*
4. *if* $\bar{y}_2 \neq 0$, *then* $\lim_{x\to\bar{x}}(f/g)(x) = \bar{y}_1/\bar{y}_2$.

Proof: Let us prove only the first two statements of the theorem. For the first statement, we have that for all $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that $|f(x) - \bar{y}_1| < \varepsilon/2$ for all $x \in B'_{\delta_1}(\bar{x}) \cap X$, and $|g(x) - \bar{y}_2| < \varepsilon/2$ for all $x \in B'_{\delta_2}(\bar{x}) \cap X$. Let $\delta = \min\{\delta_1, \delta_2\} > 0$. Then, by construction, for all $x \in B'_\delta(\bar{x}) \cap X$ we have that $|f(x) - \bar{y}_1| < \varepsilon/2$ and $|g(x) - \bar{y}_2| < \varepsilon/2$, which implies, by triangle inequality, that

$$|(f + g)(x) - (\bar{y}_1 + \bar{y}_2)| \leq |f(x) - \bar{y}_1| + |g(x) - \bar{y}_2| < \varepsilon.$$

For the second statement, note first that if $\alpha = 0$ the proof is trivial. Then, consider $\alpha \neq 0$. Since $\lim_{x\to\bar{x}} f(x) = \bar{y}_1 \in \mathbb{R}$, then for all $\varepsilon > 0$, there is some $\delta > 0$ such that, for all $x \in B'_\delta(\bar{x}) \cap X$, $|f(x) - \bar{y}_1| < \varepsilon/|\alpha|$. This implies that

$$|(\alpha f)(x) - \alpha\bar{y}_1| = |\alpha(f(x) - \bar{y}_1)| = |\alpha||f(x) - \bar{y}_1| < \varepsilon,$$

and, therefore, that $\lim_{x\to\bar{x}}(\alpha f)(x) = \alpha\bar{y}_1$. *Q.E.D.*

Given the relationship found in Theorem 6 , it comes as no surprise that a theorem analogous to the previous one holds for sequences.

[2] The following notation is introduced. We define $(f + g) : X \to \mathbb{R}$ by $(f + g)(x) = f(x) + g(x)$. We define $(f.g)$ and (αf), for $\alpha \in \mathbb{R}$, accordingly. Now, define $X_g^* = \{x \in X | g(x) \neq 0\}$. Then, we define $(\frac{f}{g}) : X_g^* \to \mathbb{R}$ by $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$.

Theorem 8. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences in $\mathbb{R}$. Suppose that for numbers $a, b \in \mathbb{R}$, we have that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then,*

1. $\lim_{n\to\infty}(a_n + b_n) = a + b$;
2. $\lim_{n\to\infty}(\alpha a_n) = \alpha a$, *for all* $\alpha \in \mathbb{R}$;
3. $\lim_{n\to\infty}(a_n.b_n) = a.b$; *and*
4. *if* $b \neq 0$ *and* $b_n \neq 0$ *for all* $n \in \mathbb{N}$, *then* $\lim_{n\to\infty}(a_n/b_n) = a/b$.

The proof of the first two parts is left as an exercise. The following theorem is also very useful:

Theorem 9. *For sequences $(a_n)_{n=1}^{\infty}$ (in $\mathbb{R}$) such that $a_n > 0$ for all $n \in \mathbb{N}$, the following equivalence holds:*

$$\lim_{n\to\infty} a_n = \infty \Leftrightarrow \lim_{n\to\infty} \frac{1}{a_n} = 0.$$

Proof: Let us prove the sufficiency statement, leaving necessity as an exercise. Suppose that $\lim_{n\to\infty}(1/a_n) = 0$ and fix $\Delta > 0$. Then, for some $n^* \in \mathbb{N}$ one has that $|1/a_n - 0| < 1/\Delta$ when $n \geq n^*$; since each $a_n > 0$, it follows that $a_n > \Delta$. *Q.E.D.*

Exercise 8. *Repeat the last part of Exercise 2, using the previous theorem. Is it easier? Show that*

$$\lim_{n\to\infty}\left(\frac{15n^5 + 73n^4 - 118n^2 - 98}{30n^5 + 19n^3}\right) = \frac{1}{2}.$$

A very useful property of limits (for both sequences and functions) is that they preserve weak inequalities. This is the content of the following theorem, whose proof is left as an exercise.

Theorem 10. *Consider a sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$ and a number $a \in \mathbb{R}$. If $a_n \leq \alpha$, for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = a$, then $a \leq \alpha$. Similarly, if $a_n \geq \alpha$, for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = a$, then $a \geq \alpha$.*

Exercise 9. *Can we strengthen our results to say: "Consider a sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$ and a number $a \in \mathbb{R}$. If $a_n < \alpha$, for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = a$, then $a < \alpha$."?*

The next result is the counterpart for limits of functions; again, the proof is left as an exercise.

Theorem 11. *Consider $f : X \to \mathbb{R}$ and $\bar{y} \in \mathbb{R}$, and let $\bar{x} \in \mathbb{R}^K$ be a limit point of X. If $f(x) \leq \gamma$ for all $x \in X$, and $\lim_{x\to\bar{x}} f(x) = \bar{y}$, then $\bar{y} \leq \gamma$. Similarly, if $f(x) \geq \gamma$ for all $x \in X$, and $\lim_{x\to\bar{x}} f(x) = \bar{y}$, then $\bar{y} \geq \gamma$.*

Exercise 10. *The previous theorem can be proved by two different arguments. Can you give them both? (Hint: one argument is by contradiction; the other one uses Theorem 10 directly.)*

Corollary 1. *Consider $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$, let $\bar{y}_1, \bar{y}_2 \in \mathbb{R}$, and let $\bar{x} \in \mathbb{R}^K$ be a limit point of X. If $f(x) \geq g(x)$, for all $x \in X$, $\lim_{x\to\bar{x}} f(x) = \bar{y}_1$ and $\lim_{x\to\bar{x}} g(x) = \bar{y}_2$, then $\bar{y}_1 \geq \bar{y}_2$.*

Obviously, a sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}^K$ is nothing but an array of K sequences in $\mathbb{R}$: sequence $(a_{k,n})_{n=1}^{\infty}$ for each $k = 1, \ldots, K$. So, it should not come as no surprise that some relations exist between these objects.

THEOREM 12. *Sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}^K$ is bounded if, and only if, for each $k = 1, \ldots, K$, sequence $(a_{k,n})_{n=1}^{\infty}$ in $\mathbb{R}$ is bounded.*

THEOREM 13. *Sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}^K$ converges to a if, and only if, for each $k = 1, \ldots, K$, sequence $(a_{k,n})_{n=1}^{\infty}$ in $\mathbb{R}$ converges to a_k.*

Proof: Let us prove sufficiency first. Given any $\epsilon > 0$, for each k there is some $n_k^* \in \mathbb{N}$ such that $|a_{k,n} - a_k| < \epsilon/\sqrt{K}$ whenever $n \geq n_k^*$. Letting $n^* = \max\{n_1^*, \ldots, n_K^*\} \in \mathbb{N}$ and $n \geq n^*$, by construction,

$$\|a_n - a\| = (\sum_k (a_{k,n} - a_k)^2)^{1/2} < (\sum_k \epsilon^2/K)^{1/2} = \epsilon.$$

For necessity, fix k and let $\epsilon > 0$. By assumption, there is $n^* \in \mathbb{N}$ after which $\|a_n - a\| < \epsilon$, which suffices to imply that $|a_{k,n} - a_k| < \epsilon$. *Q.E.D.*

5 Topology of $\mathbb{R}^K$

From now on, we deal only with subsets of $\mathbb{R}^K$, for a finite number K; that is, whenever we introduce sets X or Y, we assume that $X, Y \subseteq \mathbb{R}^K$ and use all the algebraic structure of $\mathbb{R}^K$. We also use the structure induced in $\mathbb{R}^K$ by the Euclidean norm. Whenever we take complements, they are relative to $\mathbb{R}^K$.

5.1 Open sets

The two key concepts are those of open and closed sets.

DEFINITION. *Set X is* open *if for all $x \in X$, there is some $\varepsilon > 0$ for which $B_\varepsilon(x) \subseteq X$.*

EXAMPLE 4 (Open intervals are open sets in $\mathbb{R}$). *We define an open interval, denoted (a, b),*[3] *where $a, b \in \mathbb{R}$, as $\{x \in \mathbb{R} | a < x < b\}$. To see that these are open sets (in $\mathbb{R}$), take $x \in (a, b)$, and define $\varepsilon = \min\{x - a, b - x\}/2 > 0$. By construction, $B_\varepsilon(x) \subseteq X$. As a consequence, notice that open balls are open sets in $\mathbb{R}$. The same is true in $\mathbb{R}^K$, for any K.*

It is easy to see that if we extend the definition of the open interval (a, b) to $\{x \in \mathbb{R} | a < x < b\}$ where $a, b \in \mathbb{R} \cup \{+\infty, -\infty\}$, then it continuous to be true that open intervals are open sets. The following theorem is a specific instance of a more general principle: in any space, the empty set and the universe are open sets.

THEOREM 14. *The empty set and $\mathbb{R}^K$ are open.*

Proof: A set X fails to be open if one can find $x \in X$ such that for all $\varepsilon > 0$ one has that $B_\varepsilon(x) \cap X^c \neq \varnothing$. Clearly, $\varnothing$ cannot exhibit such property. The argument that $\mathbb{R}^K$ is open is left as an exercise. *Q.E.D.*

[3] Sometimes open intervals are denoted by $]a, b[$ rather that (a, b) in order to distinguish them from two-element sequences. We will, however, follow the more standard notation.

THEOREM 15. *The union of any collection of open sets is an open set. The intersection of any finite collection of open sets is an open set.*

Proof: For the first statement, suppose that Z is the union of a given collection of open sets (whether finite or infinite doesn't matter), and suppose that $x \in Z$. By definition, then, there exists a member X of the collection of sets such that $x \in X$. By assumption, X is open, so that $\exists \varepsilon > 0$ for which $B_\varepsilon(x) \subseteq X$, and it follows, then, that $B_\varepsilon(x) \subseteq Z$.

For the second part, suppose that Z is the intersection of a finite collection of open sets, say $\{X_1, X_2, \ldots, X_{n^*}\}$, and suppose that $x \in Z$. By definition, then, for each $n = 1, 2, \ldots, n^*$, it is true that $x \in X_n$. By assumption, each X_n is open, so that there exists $\varepsilon_n > 0$ such that $B_{\varepsilon_n}(x) \subseteq X_n$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n^*}\} > 0$. By construction, for each n, we have that $B_\varepsilon(x) \subseteq B_{\varepsilon_n}(x) \subseteq X_n$ and therefore $B_\varepsilon(x) \subseteq Z$. *Q.E.D.*

We say that point x is *an interior point of the set* X, if there is some $\varepsilon > 0$ for which $B_\varepsilon(x) \subseteq X$. The set of all the interior points of X is called *the interior of* X, and is usually denoted $\text{int}(X)$.[4] Note that $\text{int}(X) \subseteq X$.

EXERCISE 11. *Show that for every* X, $\text{int}(X)$ *is open and that* X *is open if, and only if,* $\text{int}(X) = X$.

EXERCISE 12. *Prove the following: "If* $x \in \text{int}(X)$*, then* x *is a limit point of* X*."*

EXERCISE 13. *Did we really need finiteness in the second part of Theorem 15? Consider the following infinite collection of open intervals: for all* $n \in \mathbb{N}$*, define* $I_n = (-\frac{1}{n}, \frac{1}{n})$*. Find the intersection of all those intervals, denoted* $\cap_{n=1}^{\infty} I_n$*. Is it an open set?*

5.2 Closed sets

DEFINITION. *Set* X *is* closed *if for every sequence* $(x_n)_{n=1}^{\infty}$ *in* $\mathbb{R}^K$ *that satisfies that* $x_n \in X$*, at all* $n \in \mathbb{N}$*, and for which there is some* $\bar{x} \in \mathbb{R}^K$ *to which it converges, we have that* $\bar{x} \in X$*.*

Given a set $X \subseteq \mathbb{R}^K$, we define its *closure*, denoted by $\text{cl}(X)$, as the set[5]

$$\text{cl}(X) = \{x \in \mathbb{R}^K | \forall \varepsilon > 0, B_\varepsilon(x) \cap X \neq \varnothing\}.$$

As before, the empty set and the universe are closed sets. In $\mathbb{R}^K$ these are the only two sets that have both properties, but this principle does not generalize to other spaces.

THEOREM 16. *The empty set and* $\mathbb{R}^K$ *are closed.*

Proof: In order for set X to fail to be closed, there has to exist $(x_n)_{n=1}^{\infty}$ satisfying that all $x_n \in X$, and that $x_n \to \bar{x}$, yet $\bar{x} \notin X$. Clearly, one cannot find such sequence if $X = \varnothing$. The argument that $\mathbb{R}^K$ is left as an exercise. *Q.E.D.*

EXERCISE 14. *If* $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$*,* $a < b$*, is* (a, b) *closed? We define the* half-closed *interval* $(a, b]$*, where* $a \in \mathbb{R} \cup \{-\infty\}$*,* $b \in \mathbb{R}$*,* $a < b$*, as* $(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$*. Similarly, we define the half-closed interval* $[a, b)$ *where* $a \in \mathbb{R}$*,* $b \in \mathbb{R} \cup \{\infty\}$*,* $a < b$*, as* $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$*. Are half-closed intervals closed sets? Are they open? If* $x \in \mathbb{R}^K$*, is* $\{x\}$ *an open set, a closed set or neither?*

[4] Alternative, but usual, notation is X^o.

[5] Alternative notation is $\bar{X}$.

Theorem 17. *A set X is closed if, and only if, X^c is open.*

Proof: Suppose that X^c is open, and consider any sequence $(x_n)_{n=1}^{\infty}$ satisfying that all $x_n \in X$ and converging to some $\bar{x}$; we need to show that $\bar{x} \in X$. In order to argue by contradiction, suppose that $\bar{x} \in X^c$. Since X^c is open, there is some $\varepsilon > 0$ for which $B_{\varepsilon}(\bar{x}) \subseteq X^c$. Since $x_n \to \bar{x}$, there is $n^* \in \mathbb{N}$ such that $\|x_n - \bar{x}\| < \varepsilon$ when $n \geq n^*$. Then, for any $n \geq n^*$, we have that $x_n \in B_{\varepsilon}(\bar{x}) \subseteq X^c$, which is impossible.

Suppose now that X is closed, and fix $x \in X^c$. We need to show that for some $\varepsilon > 0$ one has that $B_{\varepsilon}(x) \subseteq X^c$. Again, suppose not: for all $\varepsilon > 0$, it is true that $B_{\varepsilon}(x) \cap X \neq \varnothing$. Clearly, then, for all $n \in \mathbb{N}$ we can pick $x_n \in B_{1/n}(x) \cap X$. Construct a sequence $(x_n)_{n=1}^{\infty}$ of such elements. Since $1/n \to 0$ it follows that $x_n \to x$, and all $x_n \in X$ and X is closed, then $x \in X$, contradicting the fact that $x \in X^c$. *Q.E.D.*

Theorem 18. *The intersection of* any *collection of closed sets is closed. The union of any finite collection of closed sets is closed.*

Proof: Left as an exercise. (Hint: do you remember DeMorgan's Laws?) *Q.E.D.*

Exercise 15. *Prove the following: "Given a set $X \subseteq \mathbb{R}^K$, one has $x \in cl(X)$ if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$ in X such that $x_n \to x$."*

Exercise 16. *Prove the following: "For every set $X \subseteq \mathbb{R}^K$, $X \subseteq cl(X)$, and X is closed if, and only if, $X = cl(X)$."*

Example 5. *Closed intervals are closed sets. We define an closed interval, denoted $[a, b]$, where $a, b \in \mathbb{R}$ and $a \leq b$ as $\{x \in \mathbb{R} | a \leq x \leq b\}$. To see that these are closed sets, notice that $[a, b]^c = (-\infty, a) \cup (b, \infty)$, and conclude based on previous results.*

Exercise 17. *Did we really need finiteness in the second part of Theorem 18? Consider the following infinite collection of closed intervals: for all $n \in \mathbb{N}$, define $J_n = [1 + \frac{1}{n}, 3 - \frac{1}{n}]$. Find the union of all those intervals, denoted $\cup_{n=1}^{\infty} J_n$. Is it a closed set?*

Exercise 18. *Given $X \subseteq \mathbb{R}^K$, define* the boundary of X *as* $\text{bd}(X) = \text{cl}(X) \setminus \text{int}(X)$. *Prove the following statements:* "X is closed if, and only if, $\text{bd}(X) \subseteq X$. It is open if, and only if, $\text{bd}(X) \cap X = \varnothing$." *Also, prove that*

$$\text{bd}(X) = \{x \in \mathbb{R}^K | \forall \varepsilon > 0, B_{\varepsilon}(x) \cap X \neq \varnothing \textit{ and } B_{\varepsilon}(x) \cap X^c \neq \varnothing\}.$$

5.3 Compact sets

A set $X \subseteq \mathbb{R}^K$ is said to be *bounded above* if there exists $\alpha \in \mathbb{R}^K$ such that $x \leq \alpha$ for all $x \in X$; it is said to be *bounded below* if for some $\beta \in \mathbb{R}^K$ one has that $x \geq \beta$ is true for all $x \in X$; and it is said to be *bounded* if it is bounded above and below.

Exercise 19. *Show that a set X is bounded if and only if there exists $\alpha \in \mathbb{R}_+$ such that $\|x\| \leq \alpha$ for all $x \in X$.*

Definition. *A set $X \subseteq \mathbb{R}^K$ is said to be* compact *if it is closed and bounded.*

Exercise 20. *Prove the following statement: if $(x_n)_{n=1}^{\infty}$ is a sequence defined on a compact set X, then it has a subsequence that converges to a point in X.*

6 Continuity

6.1 Continuous functions

DEFINITION. *Function $f : X \to \mathbb{R}$ is* continuous at $\bar{x} \in X$ *if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(\bar{x})| < \varepsilon$ for all $x \in B_\delta(\bar{x}) \cap X$. It is* continuous *if it is continuous at all $\bar{x} \in X$.*

Note that continuity at $\bar{x}$ is a local concept. Second, note that $\bar{x}$ in the definition may but need not be a limit point of X. Therefore, two points are worth noticing: if $\bar{x}$ is not a limit point of X, then any $f : X \to \mathbb{R}$ is continuous at $\bar{x}$ (why?); and if, on the other hand, $\bar{x}$ is a limit point of X, then $f : X \to \mathbb{R}$ is continuous at $\bar{x}$ if, and only if, $\lim_{x \to \bar{x}} f(x) = f(\bar{x})$. Intuitively, this occurs when a function is such that in order to get arbitrarily close to $f(\bar{x})$ in the range, all we need to do is to get close enough to $\bar{x}$ in the domain. By Theorem 6, it follows that when $\bar{x} \in X$ is a limit point of X, f is continuous at $\bar{x}$ if, and only if, whenever we take a sequence of points in the domain that converges to $\bar{x}$, the sequence formed by their images converges to $f(\bar{x})$ (that in this case the concept is not vacuous follows from Exercise 5).

EXERCISE 21. *Consider the function introduced in Exercise 7. Is it continuous?*

EXERCISE 22. *Consider the function introduced in Example 3. Is it continuous? What if we change the function, slightly, as follows: $f : \mathbb{R} \to \mathbb{R}$, defined as*

$$f(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -1, & \text{if } x < 0. \end{cases}$$

Is it continuous?

6.2 Properties and the Intermediate Value Theorem

The following properties of continuous functions are derived from the properties of limits. They are all very useful in economics.

THEOREM 19. *Suppose that $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are continuous at $\bar{x} \in X$, and let $\alpha \in \mathbb{R}$. Then, the functions $f + g$, αf and $f \cdot g$ are continuous at $\bar{x}$. Moreover, if $g(\bar{x}) \neq 0$, then $\frac{f}{g}$ is continuous at $\bar{x}$.*

THEOREM 20. *Fucntion $f : \mathbb{R}^K \to \mathbb{R}$ is continuous if and only if, for every open set $U \subseteq \mathbb{R}$, the set $f^{-1}[U]$ is open.*

Proof: Fix $\bar{x} \in \mathbb{R}^K$ and $\varepsilon > 0$. By Example 4, we know that $B_\varepsilon(f(\bar{x}))$ is open and, therefore, so is $f^{-1}[B_\varepsilon(f(\bar{x}))]$. Since $\bar{x} \in f^{-1}[B_\varepsilon(f(\bar{x}))]$, we have that there exists some $\delta > 0$ for which $B_\delta(\bar{x}) \subseteq f^{-1}[B_\varepsilon(f(\bar{x}))]$. For such δ, the latter means that that for all $x \in B_\delta(\bar{x})$ one has that $|f(x) - f(\bar{x})| < \varepsilon$.

Now, let $U \subseteq \mathbb{R}$ be an open set, and let $\bar{x} \in f^{-1}[U]$. By definition, $f(\bar{x}) \in U$, and since U is open, there is some $\varepsilon > 0$ for which $B_\varepsilon(f(\bar{x})) \subseteq U$. Since f is continuous, there exists $\delta > 0$ such that $|f(x) - f(\bar{x})| < \varepsilon$ for all $x \in B_\delta(\bar{x})$. The latter implies $B_\delta(\bar{x}) \subseteq f^{-1}[U]$. *Q.E.D.*

THEOREM 21 (The Intermediate Value Theorem in $\mathbb{R}$). *If function $f : [a, b] \to \mathbb{R}$ is continuous, then for every number γ between $f(a)$ and $f(b)$ there exists an $x \in [a, b]$ for which $f(x) = \gamma$.*[6]

[6] It does not matter whether $f(a) \geq f(b)$ or $f(a) < f(b)$ – we could simply have written that $\gamma \in [f(a), f(b)] \cup [f(b), f(a)]$.

6.3 Left- and Right-continuity

Consider a function $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$. Suppose that $\bar{x}$ is a limit point of X, and let $\ell \in \mathbb{R}$. One says that $\lim_{x \searrow \bar{x}} f(x) = \ell$, when for every $\varepsilon > 0$ there is a number $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ whenever $x \in X \cap B_\delta(\bar{x})$ and $x > \bar{x}$. In such case, function f is said to converge to ℓ as x tends to $\bar{x}$ from above. Similarly, $\lim_{x \nearrow \bar{x}} f(x) = \ell$, when for every $\varepsilon > 0$ there is $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ for all $x \in X \cap B_\delta(\bar{x})$ satisfying that $x > \bar{x}$. In this case, f is said to converge to ℓ as x tends to $\bar{x}$ from below.

Function $f : X \to \mathbb{R}$ is *right-continuous at* $\bar{x} \in X$, where $\bar{x}$ is a limit point of X, if $\lim_{x \searrow \bar{x}} f(x) = f(\bar{x})$. It is *right-continuous* if it is right-countinuous at every $\bar{x} \in X$ that is a limit point of X. Similarly, $f : X \to \mathbb{R}$ is *left-continuous at* $\bar{x}$ if $\lim_{x \nearrow \bar{x}} f(x) = f(\bar{x})$, and one says that f is left-continuous if it is left-continuous at all limit point $\bar{x} \in X$.

EXERCISE 23. *Consider the function introduced in Exercise 22. Is it right-continuous? Left-continuous? What if, keeping the rest of the function unchanged, we redefine $f(0) = -1$? Is it left- or right-continuous? What if $f(0) = 1$.*