

# Self-Organisation of magnetoacoustic waves in a thermally unstable environment

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The nonlinear evolution of magnetoacoustic waves in a nonadiabatic plasma are investigated analytically. The effect of plasma activity due to linear and quadratic heating and radiative cooling on propagating magnetoacoustic waves in a uniform plasma are considered. A nonlinear evolution equation is derived and stationary solutions are looked for the various combination of signs of the linear and quadratic heating-cooling terms, which determine the thermal activity of the plasma. It is shown that self-organizing magnetoacoustic waves (autowaves) exist in an active plasma. These wave have amplitudes that are independent from the initial conditions and function of plasma properties only. Their potential diagnostic purposes are discussed. Furthermore, magnetoacoustic auto-solitary waves are shown to exist. They have been modeled using a novel perturbative technique which allows to determine their propagation speed and shape.

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## I. INTRODUCTION

The study of the propagation and stability of magnetohydrodynamic (MHD) waves in nonadiabatic plasmas is important for the understanding of the formation and evolution of structuring at various spatial and temporal scales. Nonadiabaticity in the form of optically thin radiation and plasma heating provides free energy for a wave to be amplified. We call this process plasma activity. Radiative losses from optically thin plasmas depend in a complex manner on the thermodynamic properties of the plasma [1, 2]. Equally, expressions for plasma heating are highly dependent on application and the physical situation. However, valuable insight can be gained by studying generic heating-cooling functions in the energy balance. It is known that a negative gradient of this function w.r.t. thermodynamic quantities leads to a thermal instability [3]. This instability has been extensively studied in various contexts for the formation of cool condensations such as solar prominence formation [4], interstellar clouds

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[5] and edge phenomena in tokamaks [6]. Most studies focused on the linear stage of the instability [7, 8]. One-dimensional nonlinear studies have also been performed numerically [9–12] and the possibility for the appearance of steady solutions i.e. auto-solitons, was shown by [13]

Here, we shall study the nonlinear dynamics of magnetoacoustic waves in a thermally active, linearly dispersionless dissipative plasma analytically. The generic nonlinear evolutionary equation describing dynamic processes in an active medium including self-organization is a quasilinear and nonlinear parabolic equation

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{F}(\nabla \mathbf{u}, \mathbf{u}) + \nabla \cdot (\mathbf{D} \cdot \nabla \mathbf{u}) + \mathbf{A}(r, t, \mathbf{u}), \quad (1)$$

where  $\mathbf{u}$  is the physical quantity describing the system. Such an equation often occurs in for example hydrodynamics, nonlinear optics or chemical reactions [14]. The three terms on the right-hand-side describe nonlinearity, diffusion and activity, respectively. The quantity  $\mathbf{A}$  i.e. amplification, may be of a form that allows wave excitation. The simplest form of  $\mathbf{A}$  that can achieve this is  $\mathbf{A} = \mathbf{u}$ . Here, the dependency of the heating and radiative cooling functions on thermodynamics quantities determine the form of  $\mathbf{A}$ .

The influence of activity on shock formation of MHD waves has been studied by [15] using a linear activity function related to the gradient of the heating-cooling function. They showed that a shock may form faster or be completely suppressed depending on the sign of the activity. Furthermore, [16] investigated how linear activity leads to the existence of a slow magnetoacoustic autowave solution. An autowave is an example of a self-excited nonlinear wave. With the amplitude that is fully prescribed by the plasma and is independent from the initial conditions.

The heating-cooling function is in most cases not monotonic and often a turnover of the radiative loss function from a positive slope to a negative slope with increasing temperature is observed. Therefore, to model the effect of local extrema in the heating-cooling, we have chosen to consider a generic form of activity that includes a quadratic nonlinear term, i.e.  $A = au + bu^2$ . As we shall see, this extension introduces new physics such as the existence of solitary waves.

The paper is structured as follows. In Sect. I the physical model used is introduced and the nonlinear evolutionary equation governing propagating magnetoacoustic waves in an active plasma is derived. In Sect. II B The classes and stability of stationary solutions are examined, making use of perturbation theory near fixed points in phase space. In Sect. III the autowave solution is examined in detail. In Sect. IV the auto-solitary wave is studied using a novel perturbation technique.

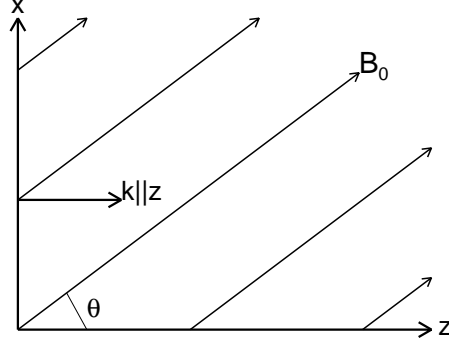


figure. 1: Model geometry with the equilibrium magnetic field,  $B_0$ , and direction of wave propagation

## II. MODEL AND GOVERNING EQUATIONS

### A. Equilibrium and governing MHD equations

The nonlinear MHD waves are studied in a uniform plasma for which a Cartesian coordinate system  $x, y, z$  is adopted. The equilibrium magnetic field,  $B_0$ , lies in the  $xz$ -plane. It has a constant magnitude  $B_0$  and angle  $\theta$  with respect to the  $z$ -axis. The wave propagation is fixed to be in the  $z$ -direction. Hence, a value of  $\theta = 0, \pi$  ( $\theta = \pm\pi/2$ ) represents parallel (perpendicular) wave propagation with respect to the magnetic field direction. The model geometry is illustrated in Figure 1. Equilibrium flows are not considered. The typical speeds in the model are the Alfvén speed,  $C_A = B_0 / \sqrt{\mu_0 \rho_0}$ , and the sound speed,  $C_S = \sqrt{\gamma p_0 / \rho_0}$ , where  $p_0, \rho_0, \gamma$  and  $\mu_0$  are the equilibrium pressure and density, the ratio of specific heats which, will be taken as 5/3, and the permittivity of free space, respectively. In what follows we shall use the notations  $C_{Ax} = C_A \sin \theta$  and  $C_{Az} = C_A \cos \theta$ .

The following version of the MHD equations are used:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \quad (2)$$

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}, \quad (3)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}), \quad (4)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (5)$$

$$\frac{dp}{dt} - \frac{\gamma p}{\rho} \frac{d\rho}{dt} = (\gamma - 1) [\mathcal{L}(\rho, p) + \nabla \cdot (\kappa \nabla T)], \quad (6)$$

where  $\mathbf{V}$ ,  $\mathbf{B}$ ,  $p$ ,  $\rho$  and  $T$  are the plasma velocity, magnetic field, pressure, density and temperature, respectively. The later three quantities are related through the ideal gas law. We consider MHD waves with small amplitudes such that they can be

described using an expansion procedure of the MHD equations in wave amplitude. The plasma quantities are expanded around the equilibrium as  $f(z, t) = f_0 + f'(z, t)$ , where  $f'$  now denotes a perturbed quantity. It is assumed that perturbations vary spatially only in the  $z$ -direction and hence are plane.

The MHD equations used here include in the energy balance the effects of non-specified heating, thermal conduction and radiative losses. These processes are considered weak such that perturbations of these terms are at least of the order of the square of the wave amplitude. Viscous and resistive processes have been ignored. The first term on the right-hand side of Eq. (6),  $\mathcal{L}(\rho, p) = \mathcal{H}(\rho, p) - L_r(\rho, p)$  combines the effects of heating and radiative cooling of the plasma. The heating  $\mathcal{H}(\rho, p)$  is an arbitrary function of density and pressure and may be prescribed depending on the specific physical scenario of interest. The radiative cooling function  $L_r(\rho, p)$  is due to optically thin radiation and depends on the details of radiation profiles of line and free emission from minority species in the plasma [1, 2]. Therefore, both heating and radiative losses may be a complicated function of density and pressure. A radiative loss function that decreases with increasing temperature is known to be a main driver of thermal instabilities [3]. This possible scenario corresponds here to a positive slope of  $\mathcal{L}$ . We expand  $\mathcal{L}(\rho, p)$  in a Taylor expansion up to second order around the equilibrium values  $\rho_0$  and  $p_0$ :

$$\begin{aligned} \mathcal{L}(\rho, p) \approx & \mathcal{L}(\rho_0, p_0) + \frac{\partial \mathcal{L}}{\partial \rho} \rho' + \frac{\partial \mathcal{L}}{\partial p} p' \\ & + \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial \rho^2} \rho'^2 + \frac{\partial^2 \mathcal{L}}{\partial \rho \partial p} \rho' p' + \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial p^2} p'^2, \end{aligned} \quad (7)$$

where all derivatives are evaluated at the equilibrium density and pressure. The inclusion of the second order derivatives allows to describe the role of heating and radiative cooling processes close to and at extrema in the heating-cooling function. To take into account these possibilities, we consider in the further analysis that the terms in Eq. (7) involving the first and second order derivatives are of the same order, i.e. quadratic in the wave amplitude.

The second term of Eq. (6) represents thermal diffusion where the thermal conduction coefficient  $\kappa$  is of the form

$$\kappa = \kappa_{\parallel} \cos^2 \theta + \kappa_{\perp} \sin^2 \theta, \quad (8)$$

where  $\kappa_{\parallel}$  and  $\kappa_{\perp}$  are the thermal conduction coefficients parallel and perpendicular to the magnetic field in the  $x$ - $z$  plane. In classical Braginskii transport theory [17], heat conduction is much more efficient parallel to the magnetic field than across, i.e.  $\kappa_{\perp} \ll \kappa_{\parallel}$ .  $\kappa_{\parallel} = 10^{-11} T^{5/2} \text{ W m}^{-1} \text{ K}^{-1}$  where the value of proportionality is a function of fundamental constants, the ion charge state (taken to be unity) and the Coulomb logarithm (a weak function of density and temperature, assumed constant). However, in tokamak experiments, thermal conduction is anomalous and the perpendicular conduction is significantly enhanced above classical values [18].

Taking into account the equilibrium model and choice of perturbation, nonlinear equations describing the Alfvén wave and

magnetoacoustic waves can be described. In the particular model, the Alfvén wave causing perturbations  $V_y$  and  $B_y$  are decoupled from the magnetoacoustic wave and cannot be excited nonlinearly if the Alfvén wave amplitude is zero initially. Therefore, we take  $y$ -components of the perturbation to be zero. The derivation of the wave equations from the perturbed MHD equations is detailed in Appendix A. There, it is shown that the nonlinear wave equation for magnetoacoustic waves in terms of  $V_z$  is given as (where the prime as been dropped for ease of notation):

$$\begin{aligned} & \left[ \frac{\partial^4}{\partial t^4} - (C_A^2 + C_S^2) \frac{\partial^4}{\partial t^2 \partial z^2} + C_{Az}^2 C_S^2 \frac{\partial^4}{\partial z^4} \right] V_z \\ &= 2C_S^2 \left[ L_1 + K \frac{\partial^2}{\partial z^2} \right] \left[ \frac{\partial^2}{\partial t^2} - C_{Az}^2 \frac{\partial^2}{\partial z^2} \right] \int \frac{\partial^2 V_z}{\partial z^2} dt' \\ &- 2C_S^2 L_2 \left[ \frac{\partial^2}{\partial t^2} - C_{Az}^2 \frac{\partial^2}{\partial z^2} \right] \frac{\partial}{\partial z} \left[ \int \frac{\partial V_z}{\partial z} dt' \right]^2 + N, \end{aligned} \quad (9)$$

where

$$L_1 = \frac{(\gamma-1)}{2C_S^2} \left( \frac{\partial}{\partial \rho} + C_S^2 \frac{\partial}{\partial p} \right) \mathcal{L}, \quad (10)$$

$$L_2 = \frac{(\gamma-1)\rho_0}{4C_S^2} \left( \frac{\partial}{\partial \rho} + C_S^2 \frac{\partial}{\partial p} \right)^2 \mathcal{L}, \quad (11)$$

$$K = \frac{(\gamma-1)^2 \kappa \tilde{\mu}}{2\gamma \mathcal{R} \rho_0}, \quad (12)$$

$$\begin{aligned} N &= \frac{1}{\rho_0} \mathcal{D}_A \left[ \frac{\partial N_6}{\partial t} - \frac{\partial}{\partial z} (C_S^2 N_7 + N_1) \right] \\ &- \frac{B_{0x}}{\rho_0 \mu_0} \frac{\partial^2}{\partial z \partial t} \left( \frac{\partial N_2}{\partial t} + \frac{B_{0z}}{\rho_0} \frac{\partial N_4}{\partial z} \right). \end{aligned} \quad (13)$$

The left-hand side of Eq. (9) is the linear magnetoacoustic wave operator. The right-hand side collects the nonlinear and non-ideal terms.  $L_1$  and  $L_2$  contain the linear and quadratic contributions from the heating-cooling function and  $K$  contains the thermal conduction where  $\kappa_0$  denotes the thermal conduction coefficient as a function of the equilibrium quantities.  $\mathcal{R}$  and  $\tilde{\mu}$  are the gas, and atomic mass constant respectively.  $N$  is a function of the nonlinear terms  $N_j$ ,  $j = 0..7$ , which are given in Appendix A. The right-hand side are assumed to be weak.

### B. Nonlinear evolutionary equation

The evolution of the velocity component  $V_z$  operates on two scales: a fast scale following the linear propagation of the wave phase plane, and a slow scale representing the slow evolution of the wave properties due to nonlinearity and non-adiabaticity. Therefore, the following multiple scales are introduced:

$$\zeta = \epsilon^{1/2} (z - Ct), \quad \tau = \epsilon^{-3/2} t, \quad (14)$$

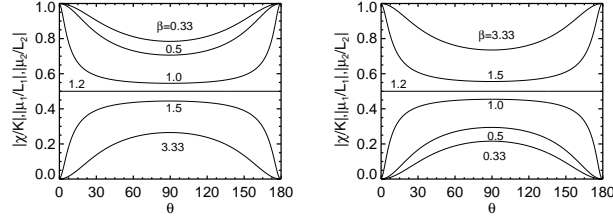


figure. 2: Non-adiabatic terms ( $|\mu_1/L_1|, |\mu_2/L_2|$ ) and thermal conduction ( $|\chi/K|$ ) as a function of angle  $\theta$  (in degrees) for the slow (left) and fast (right) magnetoacoustic modes for a range of values of the plasma- $\beta$ .

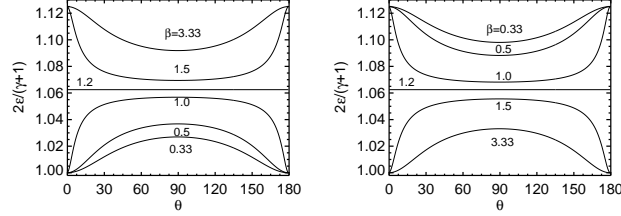


figure. 3: Nonlinearity ( $2\epsilon/(\gamma + 1)$ ) as a function of angle  $\theta$  (in degrees) for the slow (left) and fast (right) magnetoacoustic modes for a range of values of the plasma- $\beta$ .

with  $\epsilon$  a expansion parameter representing the order of a term with respect to the wave amplitude. The speed  $C$  corresponds to the slow or fast magnetoacoustic phase speed, which is given by the solutions of the dispersion relation  $C^4 - (C_A^2 + C_s^2)C^2 + C_{Az}^2 C_s^2 = 0$ . Using Eq. (14) and after a few straightforward manipulations Eq. (9) is written in the nonlinear evolutionary wave equation in a frame of reference travelling at a speed  $C$ :

$$\frac{\partial V_z}{\partial \tau} - \chi \frac{\partial^2 V_z}{\partial \zeta^2} + \epsilon V_z \frac{\partial V_z}{\partial \zeta} + \mu_1 V_z + \mu_2 V_z^2 = 0, \quad (15)$$

where the coefficients are given as

$$\chi = K \frac{C_s^2(C^2 - C_{Az}^2)}{C^2(2C^2 - C_s^2 - C_A^2)}, \quad (16)$$

$$\begin{aligned} \epsilon = & \frac{1}{2}(\gamma + 1) \frac{C_s^2(C^2 - C_{Az}^2)}{C^2(2C^2 - C_s^2 - C_A^2)} \\ & + \frac{3C^2 C_{Ax}^2}{2(C^2 - C_{Az}^2)(2C^2 - C_s^2 - C_A^2)} \end{aligned} \quad (17)$$

$$\mu_1 = -L_1 \frac{C_s^2(C^2 - C_{Az}^2)}{C^2(2C^2 - C_s^2 - C_A^2)}, \quad (18)$$

$$\mu_2 = -L_2 \frac{C_s^2(C^2 - C_{Az}^2)}{C^2(2C^2 - C_s^2 - C_A^2)}. \quad (19)$$

Eq. (15) is of the form of a generalised Burgers-Fischer Equation with  $\chi$ ,  $\epsilon$ ,  $\mu_1$  and  $\mu_2$  the magnetically modified coefficients of thermal diffusivity, nonlinearity, linear and quadratic plasma activity. A version of this equation with  $\mu_2 = 0$  was investigated

by [15, 16]. Note that  $\chi$ ,  $\mu_1$  and  $\mu_2$  (also  $\varepsilon$  for propagation parallel to the magnetic field) have the same dependency on the magnetic field. Also, in the limit of no magnetic field, the coefficients simplify to  $\chi = K$ ,  $\varepsilon = (\gamma + 1)/2$ ,  $\mu_1 = -L_1$  and  $\mu_2 = -L_2$ . Figures 2 and 3 show the dependency of the coefficients (16)-(19) as a function of the propagation angle  $\theta$  for the slow and fast magnetoacoustic waves. As previously obtained by [15], for a unique value of plasma beta,  $\beta = 1.2$ , the coefficients are independent of  $\theta$  for both slow and fast magnetoacoustic mode. The plasma beta is defined as the ratio of the gas and magnetic pressure  $\beta = \frac{2}{\gamma} C_S^2 / C_A^2$ . Also, for propagation parallel to the magnetic field and for a low  $\beta$  plasma, the non-ideal coefficients, i.e.  $\chi$ ,  $\mu_1$  and  $\mu_2$  are equal to  $K$ ,  $-L_1$  and  $-L_2$ , respectively, for the slow wave but are equal to zero for the fast wave.

Nonlinear evolutionary Eq. (15) is made dimensionless by normalising time, distance and speed according to

$$\tau^* = |\mu_1| \tau, \quad \zeta^* = \sqrt{\frac{|\mu_1|}{\chi}} \zeta, \quad V_z^* = \frac{\varepsilon}{\sqrt{\chi |\mu_1|}} V_z, \quad (20)$$

which leads to

$$\frac{\partial V_z^*}{\partial \tau^*} - \frac{\partial^2 V_z^*}{\partial \zeta^{*2}} + V_z^* \frac{\partial V_z^*}{\partial \zeta^*} + \alpha_1 V_z^* + \alpha_2 k V_z^{*2} = 0, \quad (21)$$

where  $\alpha_1$  and  $\alpha_2$  are the signs of  $\mu_1$  and  $\mu_2$ , respectively. In other words, the sign of  $\alpha_1$  and  $\alpha_2$  refers to the sign of the first and second derivatives of the heating-cooling function. For example,  $\alpha_1 = 1$ ,  $\alpha_2 = 0$  which is the Burgers Equation, corresponds to a negative slope in  $\mathcal{L}$  (e.g. positive slope of  $L_r$  if  $\mathcal{H}$  is constant). The dimensionless parameter  $k$ ,

$$k = \frac{|\mu_2| \sqrt{\chi}}{\sqrt{|\mu_1|} \varepsilon}, \quad (22)$$

determines the strength of the second derivatives of the heating-cooling function relative to the other effects involved. Near an extremum in the heating-cooling function,  $k$  will become large.

Because the propagation speed of the nonlinear wave can vary from the phase speed of the linear wave we allow for a correction to the propagation speed, called the envelope speed. Therefore, we make an additional change of frame of reference to Eq. (21) with new running coordinate  $s = \zeta^* - C_E \tau^*$  where  $C_E$  is the dimensionless envelope speed:

$$\frac{\partial V_z^*}{\partial \tau^*} - \frac{\partial^2 V_z^*}{\partial s^2} + (V_z^* - C_E) \frac{\partial V_z^*}{\partial s} + \alpha_1 V_z^* + \alpha_2 k V_z^{*2} = 0. \quad (23)$$

The solution of this nonlinear evolutionary equation is addressed analytically and numerically. Of particular interest are stationary solutions of Eq. (23), i.e. a wave that under Galilean transformations returns a non-evolving wave and hence does not depend on the slow time-scale  $\tau^*$ . We define  $\psi(s) = V_z^*(s)$  to be a stationary solution described by the nonlinear ODE:

$$\frac{d^2 \psi}{ds^2} - (\psi - C_E) \frac{d\psi}{ds} - F(\psi) = 0, \quad (24)$$

where we define  $F(\psi) = \alpha_1 \psi \left(1 + \frac{\alpha_2}{\alpha_1} k \psi\right)$  with  $C_E$  corresponding to this solution. Because  $F$  is a quadratic in  $\psi$ , the ODE (24) allows for up to two fixed points in the system, which are at  $\psi_0 = 0, -\alpha_1/\alpha_2 k$  with  $d\psi/ds=0$ . The nature of these fixed points,

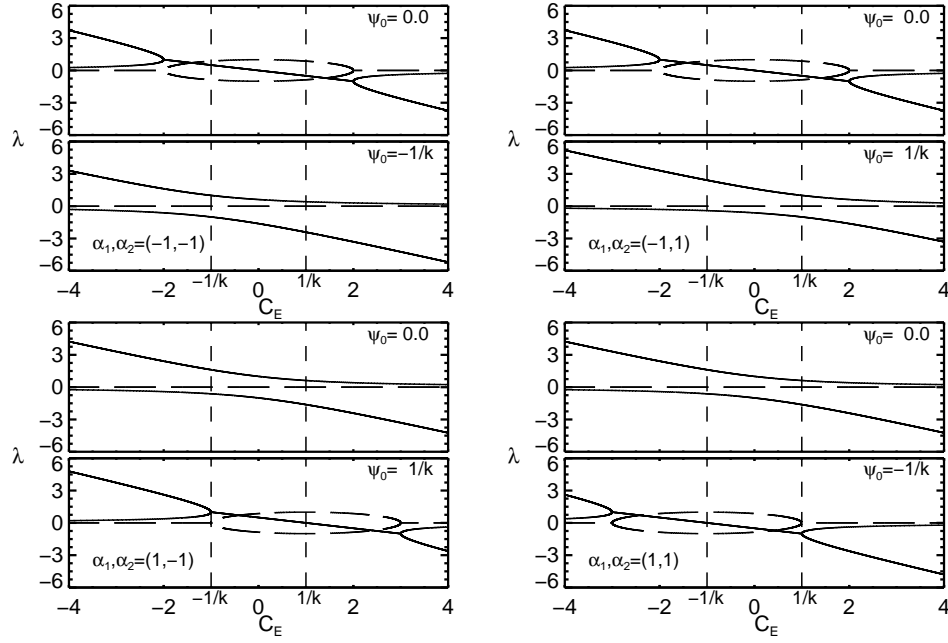


figure. 4: The real (solid) and imaginary (dashed) parts of the linear growth rate,  $\lambda$ , are shown as a function of  $C_E$  for both fixed points for  $k = 1$  and the four combinations of values of  $\alpha_1$  and  $\alpha_2$ .

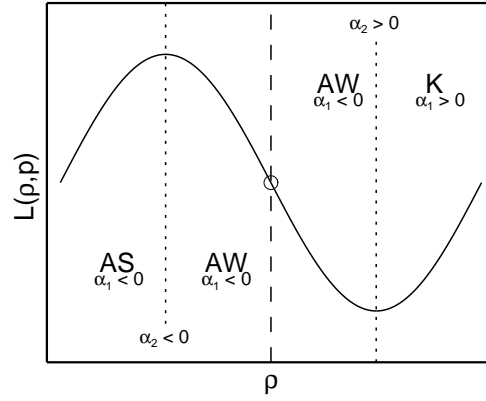


figure. 5: Heating-cooling function,  $\mathcal{L}$ , as a function of an equilibrium thermodynamic quantity for  $C_E \geq 0$ . The various types of wave solutions are highlighted (K = kink, AS = auto-solitary and AW = autowave). The circle represents the pure autowave with  $\mu_2=0$ .

and hence the stability of the wave solution near to them, is determined using linear perturbations of  $\psi$ , proportional to  $e^{\lambda s}$ , around the solution at a fixed point. It can be easily seen that the characteristic values of  $\lambda$  associated with the fixed points are given by

$$\lambda = \frac{1}{2} (\psi_0 - C_E) \pm \frac{1}{2} \sqrt{(\psi_0 - C_E)^2 - 4 \frac{dF(\psi)}{d\psi} \Big|_{\psi_0}} . \quad (25)$$



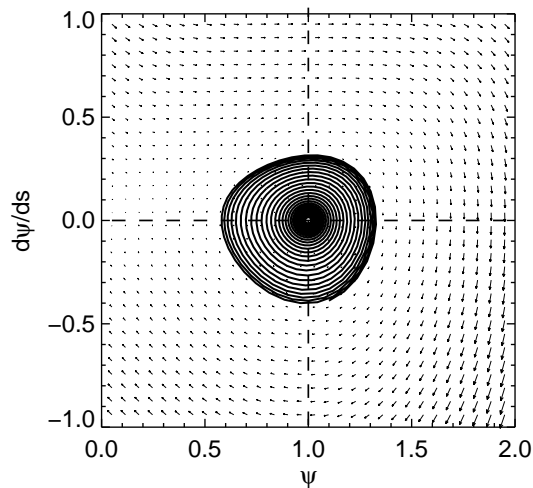


figure. 6: Normalised phase diagram  $\psi - d\psi/ds$  for  $(\alpha_1, \alpha_2) = (1, -1)$  with  $k = 1.0$  and  $C_E = 0.99$ .

The different possible solutions of  $\lambda$  for the two fixed points are shown in Figure 4 as a function of  $C_E$  for the different possible combinations of signs of  $\alpha_1$  and  $\alpha_2$ . Purely real (imaginary) values of  $\lambda$  correspond to a centre (saddle) fixed point. For  $\mu_1 < 0$  ( $\alpha_1 = -1$ ), the fixed point at  $\psi_0 = 0$  is a centre for  $C_E = 0$ . The other fixed point at  $\psi_0 = 1/\alpha_2 k$  is a saddle, whose absolute value increases as  $k$  decreases. Thus, for a linear heating-cooling function there is only a single centre fixed point. As we shall see in Sect. III, this case supports autowave solutions. For  $\mu_1 > 0$  ( $\alpha_1 = 1$ ), the fixed point at  $\psi_0 = -1/\alpha_2 k$  is a centre for  $C_E = -1/\alpha_2 k$  and only exists for finite  $k$ . The fixed point at  $\psi_0 = 0$  is a saddle. The combination of saddle and centre points is a necessary requirement for the existence of auto-solitary wave solutions, which are investigated in Sect. IV.

Figure (5) shows the magnetoacoustic stationary wave solutions for the different combinations of  $\mu_1$  and  $\mu_2$  and with  $C_E \geq 0$ . A wave that evolves to a stationary solution has properties that are independent from the initial conditions and are instead exclusively determined by the plasma itself. Therefore, measurements of stationary waves offers the opportunity of the diagnostics of non-ideal plasma effects.

### III. PERTURBATIVE SOLUTION NEAR FIXED POINT ( $|C_E| \lesssim |\psi_0|$ )

The types of solutions around the fixed points are studied using perturbation theory, where we expand  $\psi$  and  $C_E$  around  $\psi_0$ , i.e.  $\psi = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \epsilon^3\psi_3 + \dots$  and  $C_E = u_0 + \epsilon^2 u + \dots$ , and multiple-scales analysis, where we assume a fast scale  $s$  and a slow scale  $\epsilon^2 s_2$ . The order  $O(\epsilon)$  terms in Eq. (24) give the basic form of the solution around the fixed point:

$$\frac{\partial^2 \psi_1}{\partial s^2} + \sigma^2 \psi_1 = 0, \quad (26)$$

where  $\sigma^2 = -\alpha_1(1 + 2\alpha_2\psi_0/\alpha_1)$ . The fixed point  $\psi_0$ , where  $\sigma^2 = -\alpha_1$ , is a centre and gives, to first order, a bounded solution if  $\alpha_1 = -1$ . It is a saddle point for  $\alpha_1 = 1$ . This is the same result as found in the previous section. The fixed point  $\psi_0 = -\alpha_1/\alpha_2k$ , where  $\sigma^2 = \alpha_1$ , is a centre for  $\alpha_1 = 1$  and a saddle point for  $\alpha_1 = -1$ . The sign of  $\alpha_2/\alpha_1$  is not constrained. The bounded solution of Eq. (26) is written as

$$\psi_1 = A(s_2) \cos[\sigma s + \phi(s_2)] \equiv A \cos \theta, \quad (27)$$

The order  $O(\epsilon^2)$  terms in Eq. (24) give an equation

$$\frac{\partial^2 \psi_2}{\partial s^2} + \sigma^2 \psi_2 = \psi_1 \frac{\partial \psi_1}{\partial s} + \alpha_2 k \psi_1^2, \quad (28)$$

which after substituting Eq. (27) has the particular solution

$$\psi_2 = \frac{\alpha_2 k}{2\sigma^2} A^2 + \frac{1}{6\sigma} A^2 \sin 2\theta - \frac{\alpha_2 k}{6\sigma^2} A^2 \cos 2\theta. \quad (29)$$

Finally, the order  $O(\epsilon^3)$  terms in Eq. (24) give an equation

$$\frac{\partial^2 \psi_3}{\partial s^2} + \sigma^2 \psi_3 = -2 \frac{\partial^2 \psi_1}{\partial s \partial s_2} + \frac{\partial}{\partial s} (\psi_1 \psi_2) + 2\alpha_2 k \psi_1 \psi_2 - u \frac{\partial \psi_1}{\partial s}. \quad (30)$$

The terms on the right-hand side of this equation, which have the same periodicity as  $\psi_1$ , will resonantly drive the solution.

Therefore, to avoid secular behaviour these terms are set to zero, which leads to equations for the amplitude and phase:

$$\frac{dA}{ds_2} = -\frac{A}{2} \left[ u - \frac{\alpha_2 k}{4\alpha_1} \frac{\alpha_1}{\sigma} A^2 \right], \quad (31)$$

$$\frac{d\phi}{ds_2} = -\frac{A^2}{24\sigma^2} \left[ 1 + \frac{10\alpha_2 k}{\alpha_1} \frac{\alpha_1^2}{\sigma^2} \right]. \quad (32)$$

In order to have a wave solution with limit cycle behaviour, i.e. the wave amplitude returns to the same value after one period, the right-hand side of Eq. (31) must vanish. This defines the amplitude  $A$  as a function of the parameter  $k$  and the envelope speed  $C_E$ :

$$A = \pm \sqrt{\frac{4\alpha_1}{\alpha_2 k} \frac{\sigma}{\alpha_1} u} = \pm \sqrt{\frac{4\alpha_1}{\alpha_2 k} \frac{\sigma}{\alpha_1} (C_E - \psi_0)}. \quad (33)$$

For the fixed point  $\psi_0 = 0$  the amplitude is real if  $\alpha_1 = -1$  and  $\alpha_2/\alpha_1 = -1$ . Then, Eq. (33) becomes  $A = \pm 2\sqrt{C_E/k}$ . In the limit of a linear heating-cooling function, i.e.  $k$  tends to zero, we require  $C_E$  to tend to zero as well in order to obtain a finite wave amplitude.

For the fixed point  $\psi_0 = -\alpha_1/\alpha_2k$  and  $\alpha_1 = 1$  Eq. (33) becomes

$$A = \pm 2 \sqrt{\frac{\alpha_1}{\alpha_2 k} \left( C_E + \frac{\alpha_1}{\alpha_2 k} \right)}, \quad (34)$$

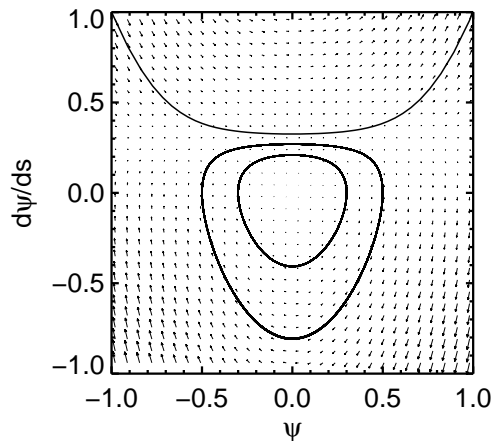


figure. 7: Normalised phase diagram  $\psi - d\psi/ds$  for the thermally unstable active regime ( $\mu_1 < 0$ ) with  $k = 0$ . Several solution curves have been shown. Closed curves correspond to oscillatory solutions.

which is real for  $C_E < 1/k$  if  $\alpha_1/\alpha_2 = -1$  and for  $C_E > -1/k$  if  $\alpha_1/\alpha_2 = 1$ .

From the solution of eq. (24) with the general nonlinear heating/cooling function a limit cycle solution arises (see Fig. 6 for an example with  $\alpha_1, \alpha_2 = 1, -1$  and  $C_E \simeq \psi_0 = k^{-1}$ ). The solution was initially near non-zero fixed point. The solution curve then grows via an unstable oscillatory process until a critical amplitude is reached. Then, the system performs oscillations with steady parameters, this is the limit cycle. The solution strongly dependent on  $C_E$ , due to the competition between  $\psi$  and  $C_E$ .

The famous Van der Pol equation is an example of similar dynamics, whereby the system undergoes oscillations that tend to a stationary state [19],[20]. When the phase dynamics evolve to a constant amplitude it defines a limit cycle. The limit cycle is interesting because, like for the Van der Pol equation, the wave characteristics such as the amplitude are independent of initial conditions but instead dependent on the system parameters only. In the following sections we shall encounter two scenarios of stationary magnetoacoustic waves that exhibit such behaviour: autowaves ( $\mu_1 < 0$ ,  $k = 0$ ) and auto-solitary waves in the presence of linear and quadratic term in the heating-cooling function ( $k \neq 0$ ).

#### IV. AUTOWAVES

Stationary solutions are sought for which the quadratic term in the heating-cooling function is ignored, i.e.  $k=0$ . Then, Eq. (24) reduces to the form

$$\frac{d^2\psi}{ds^2} - (\psi - C_E) \frac{d\psi}{ds} - \alpha_1\psi = 0 . \quad (35)$$

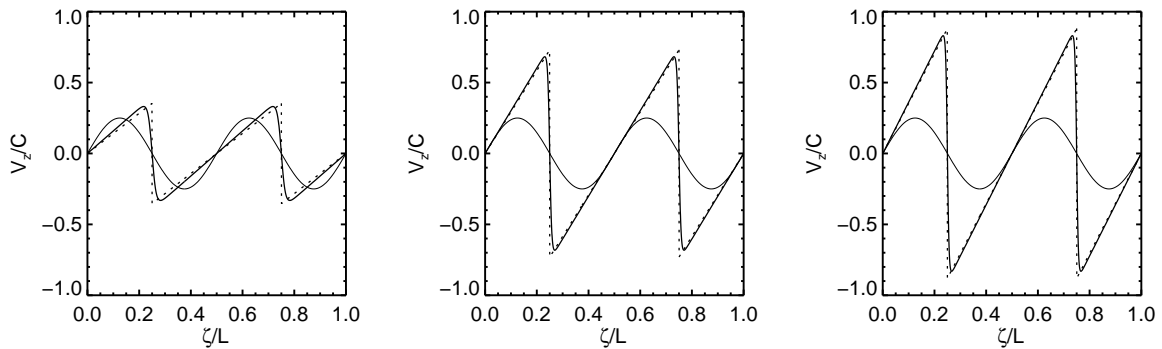


figure. 8: Velocity profile as a function of  $\zeta$  at time  $C\tau/L = 1.01$  (thick solid line) for an initial sine wave (thin solid line) for three values of  $\frac{\mu_1 L}{C} = -2.01, -4.02,$  and  $-4.82$ . Also,  $\frac{\chi}{LC} = 0.0038, \epsilon = 1.33, \mu_2 = 0, k = 0$  and  $\theta = 0$ . The dashed line is the analytic solution given by from Eq. (39).

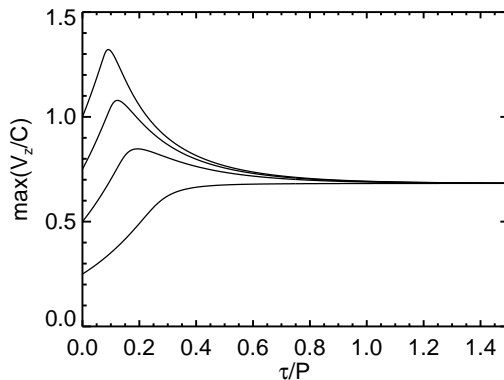


figure. 9: Maximum amplitude of  $V_z/C$  as a function of time relative to the wave period,  $\tau/P$ , for four initial amplitudes  $V_0$ . The parameters used are  $\frac{\mu_1 L}{C} = -4.6, \frac{\chi}{LC} = 0.0038, \epsilon = 1.33, k = 0$  and  $\theta = 0$ .

The  $\psi$ - $d\psi/ds$  phase-space corresponding to this equation is shown in Figure 7. Linear stability analysis round this fixed point, as shown previously, leads to the relation  $\lambda = i[C_E \pm i\sqrt{C_E^2 + 4\alpha_1}]/2$ . This shows that a solution of Eq. (35) is bounded only if  $C_E = 0$  and  $\alpha_1 = -1$ . For these conditions we have an oscillatory stationary state, which corresponds to a wave travelling at the magnetoacoustic speed  $C$  through a thermally active plasma ( $\mu_1 < 0$ ). The de-stabilizing effect of the heating-cooling function (activity) balances against the nonlinearity and dissipation to lead to the steady state. From Eq. (35) we see that condition  $d\psi/ds = 1$ , where  $d/ds(d\psi/ds) = 0$ , separates in phase space regions with bounded ( $d\psi/ds < 1$ ) from regions with unbounded solutions. Figure 7 also shows that for large amplitudes, the phase-space curve is no longer up-down symmetric. Hence, for such solutions the perturbative analysis presented in the previous section is limited.

To explore the large amplitude solutions and to demonstrate that the stationary solution can be reached from a plausible initial

perturbation and is auto-oscillatory in nature, we solve the time-dependent evolutionary equation (15) numerically. We have implemented the McCormack finite difference scheme with accuracy to second order [21]. We consider the initial condition,

$$V_z(t = 0, \zeta) = V_0 \sin\left(\frac{n\pi}{L}\zeta\right), \quad (36)$$

Where  $V_0$  and  $n$  are the initial wave amplitude and the wavenumber, respectively.  $L$  is the characteristic scale length, and also the length of the simulation interval, and is related to the wavelength as  $\lambda = 2L/n$ . At  $\zeta = 0, L$  periodic boundary conditions are imposed. In particular, for the simulations presented here, 2000 grid points have been used and two wavelengths fitted in the interval, i.e.  $n=4$ .

Figure 8 shows, for a range of values of  $\mu_1$ , the velocity profile to which an initial sinusoidal profile has evolved. For each case, the wave steepens due to nonlinearity in the plasma and forms a sawtooth-like profile. The wave is prevented from becoming multi-valued, due to the presence of thermal conduction in the system, hence a magnetoacoustic shock does not occur. Figure 9 shows that for the same plasma parameters the wave evolves to the same velocity profile and amplitude, irrespective of the initial amplitude  $V_0$ . This self-organizing property of the wave, which makes it solely dependent on plasma parameters, defines an autowave solution. For the given values of the plasma parameters, the solutions evolve to a single value within a single period ( $\tau/P < 1$ ). During the transient phase,  $0 \leq \tau/P \leq 0.6$ , the thermal activity drives up the amplitude and can even become supersonic ( $V_z/C > 1$ ). Larger velocities drive nonlinearity and the velocity profile steepens up, leading to stronger velocity gradients. Then, thermal diffusion becomes important and dampens the wave rapidly back to the subsonic range. Although the initial amplitude does not affect the final solution, it determines the rate at which the system becomes fully self-organised. The further away from the final autowave velocity amplitude we start the longer it takes to reach the final converged state.

The perturbation technique of the previous section did not allow to calculate the amplitude for the autowave. The sawtooth-like profile provides a basis function to find an analytic description of the wave. The presence of thermal conduction of course prevents the formation of the exact sawtooth. However, for the purposes here the sawtooth is a good approximation. We assume a sawtooth (triangular) profile with  $V_z = V_{z,\max} 2\zeta/\lambda$  and  $dV_z/d\zeta(\lambda/2)=0$ . Figure 8 shows that the sawtooth is a good approximation of the full solution. We integrate the stationary version of Eq. (15) (and  $C_E=0$ ),

$$\frac{d}{d\zeta} \left( \frac{dV_z}{d\zeta} - \frac{\epsilon}{2\chi} V_z^2 \right) + \frac{|\mu_1|}{\chi} V_z = 0, \quad (37)$$

over half a wavelength between  $\zeta = 0$  and  $\zeta = \lambda/2$  to obtain a condition for the maximum velocity amplitude:

$$\left[ \frac{dV_z}{d\zeta} - \frac{\epsilon}{2\chi} V_z^2 \right]_0^{\lambda/2} + \frac{|\mu_1|}{\chi} \int_0^{\lambda/2} V_z d\zeta = 0. \quad (38)$$

Using the sawtooth profile Eq. (38) reduces to

$$V_{z,\max} = \frac{\chi}{2\epsilon\lambda} \left( \frac{|\mu_1|}{\chi} \lambda^2 - 8 \right). \quad (39)$$

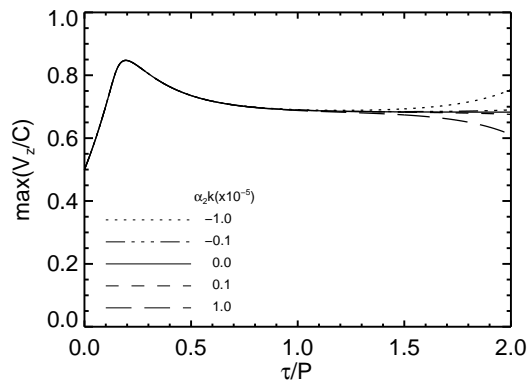


figure. 10: Maximum amplitude of  $V_z/C$  as a function of time relative to the wave period,  $\tau/P$ , for four five values of  $k$ . The initial velocity amplitude is  $V_0 = 0.5C$ . The parameters used are  $\frac{\mu_1 L}{c} = -4.02$ ,  $\frac{\chi}{LCD} = 0.0038$ ,  $\epsilon = 1.33$  and  $\theta = 0$ .

or in terms of the dimensionless amplitude  $A$ ,

$$A = \frac{1}{2\lambda^*} \left( (\lambda^*)^2 - 8 \right), \quad (40)$$

where  $\lambda^* = (|\mu_1|/\chi)^{1/2}\lambda$ . This suggest that autowaves exist for given plasma conditions only if its wavelength,  $\lambda$  is larger than  $\sqrt{8\chi/|\mu_1|}$ . Therefore, the colder or denser the plasma or the larger the slope of the heating-cooling function, the lower the threshold wavelength becomes. The preference for large wavelengths is due to the importance of thermal diffusion growing quadratically with wavenumber. Also, this threshold is a factor  $\pi/\sqrt{2}$  smaller than the wavelength threshold for an unstable plane wave for the linear terms in Eq. (15).

Thus far only a linear heating-cooling function has been considered ( $k = 0$ ). We examine the role of finite  $k$  for the existence of autowave solutions by solving numerically the time-dependent evolutionary equation (15). Figure 10 shows the maximum velocity amplitude as a function of time for several values of  $k$ . The plasma parameters are the same as for Figure 9 and  $V_0 = 0.5C$ . The early time evolution follows the linear case. However, as time progresses, the solution diverges from the final velocity. The sign of  $\alpha_2$  (sign of  $\mu_2$ ) determines the stability of the wave. For  $\alpha_2 < 0$ , the quadratic term of the heating-cooling function is negative, which added to the negative linear term, drives the solution unstable. For  $\alpha_2 > 0$ , the opposite situation occurs. The quadratic term is positive and opposes the linear term, which leads overall to damping of the wave. So, for small  $k$  the self-organizing behaviour of an autowave is preserved the longest. We name such modes quasi-linear autowaves.

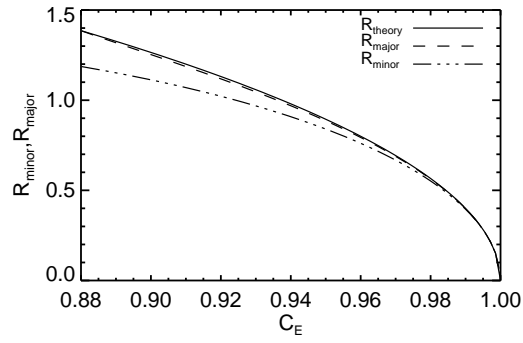


figure. 11: Major and minor amplitudes,  $R_{\text{minor}}$  and  $R_{\text{major}}$ , of an auto-solitary limit cycle as a function of  $C_E$  for  $\alpha_1, \alpha_2 = 1, -1$  and  $k=1$ . The solid line is the theoretical amplitude from the multiple-scale expansion Eq. (34).

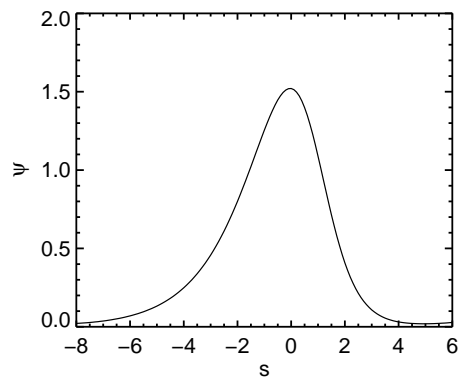


figure. 12: Plot of  $\psi$  against normalised running coordinate  $s$ , for the auto-solitary pulse, corresponding to  $(\alpha_1, \alpha_2) = (1, -1)$  with  $k=1$  and  $C_E = 0.866$ .

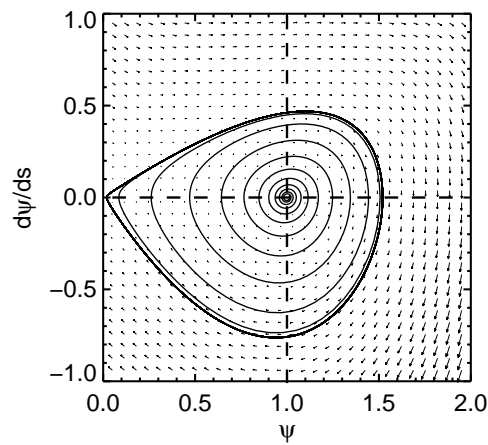


figure. 13: Normalised phase diagram  $\psi - d\psi/ds$  for  $(\alpha_1, \alpha_2) = (1, -1)$  with  $k = 1.0$  and  $C_E = 0.87$ .

## V. AUTO-SOLITARY WAVES

We consider auto-solitary wave solutions that appear with a finite quadratic term in the heating-cooling function, i.e.  $k \neq 0$ . The fixed points for Eq. (24) are  $\psi_0 = 0$  and  $\psi_0 = -\frac{\alpha_1}{\alpha_2 k}$  with  $d\psi_0/ds = 0$ . For both signs of  $\alpha_1$ , there is a pairing of a centre and saddle point. A finite value of  $C_E$  provides a driving term for instability. The presence of the second saddle point limits the growth. It is this competitive process that may lead to a possible stable state. Solving for this regime of heating-cooling we shall show that an auto-oscillatory solution exists. We restrict ourself to the consideration of the  $\mu_1 > 0$  case (see Fig. 4). However, the case  $\mu_1 < 0$  is identical. We investigate the regimes of  $C_E$  positive but much smaller than  $\psi_0 = -\alpha_1/\alpha_2 k$ . As  $C_E$  departs from  $\psi_0$  the wave profile becomes distorted. This distortion is the direct result of the solution curves interacting with both the saddle point and the centre. Such interactions are termed homoclinic bifurcation [14]. The divergence from the approximately symmetric phase-plane dynamics can be quantified by comparing the major and minor radii (amplitudes) of the limit cycle, which are defined as  $R_{\text{major}} = \psi_{\text{max}} - \psi_{\text{min}}$  and  $R_{\text{minor}} = d\psi/ds|_{\text{max}} - d\psi/ds|_{\text{min}}$ , respectively. Figure 11 shows these amplitudes as a function of  $C_E$ . The distortion becomes apparent when  $C_E$  departs from  $-\alpha_1/\alpha_2 k$ . Figure 12 shows the spatial asymmetry in  $s$  of the auto-oscillatory pulse. The pulse undergoes preferential steepening on the right side due to the competing heating/cooling terms, with thermal conduction preventing shock formation.

Figure 13 shows that for  $C_E$  far from  $-\alpha_1/\alpha_2 k$  the stationary solution of Eq. (24) is a limit cycle and takes the form of a simple closed curve when plotted in phase space (see Fig. 12). However, because in this regime, the limit cycle solution is no longer approximately symmetric, the technique of multiple-scales is no longer an accurate model. Therefore, another method has to be applied. The phase space curve suggests that the limit cycle itself satisfies an equation of the form  $(d\psi/ds)^2 + \epsilon a(\psi)(d\psi/ds) + b(\psi) = 0$  where  $\epsilon$  is an ordering parameter. We will refer to this phase-space curve as ‘‘the phase space polynomial’’.

We obtain equations for functions  $a(\psi)$  and  $b(\psi)$  in the following way. The polynomial is differentiated with respect to  $s$ . Equation (24) is used to eliminate  $d^2\psi/ds^2$ , i.e.  $d/ds(d\psi/ds) = \epsilon(\psi - C_E)(d\psi/ds) + F(\psi)$ . Thus, we find

$$\begin{aligned} & \left[ \frac{db}{d\psi} + 2F(\psi) - \epsilon^2 a \left( (\psi - C_E) + \frac{da}{d\psi} \right) \right] \left( \frac{d\psi}{ds} \right) \\ &= \epsilon b \left[ \frac{da}{d\psi} + (\psi - C_E) - \frac{a}{b} F(\psi) \right], \end{aligned} \quad (41)$$

which for a general solution  $(d\psi/ds)(\psi)$  requires that the left and right-hand side terms are zero. This yields two coupled differential equations for  $a$  and  $b$ :

$$\frac{da}{d\psi} = -2(\psi - C_E) + \frac{a}{b} F(\psi), \quad (42)$$

$$\frac{db}{d\psi} = -2F(\psi) + \epsilon^2 a \left[ \frac{a}{b} F(\psi) - (\psi - C_E) \right]. \quad (43)$$



Although Eqs. (42)-(43) are a set of two nonlinear equations, which seem more complicated than Eq. (24), it is important to note that Eqs. (42)-(43) need only to be solved around the closed limit cycle in phase-space whereas Eq. (24) must be solved over many periods to reach asymptotically the limit cycle. An analytic solution to Eqs. (42)-(43) is found perturbatively by expanding  $a$  and  $b$  in powers of the parameter  $\epsilon$ . The equations contain the term  $a\psi/b$  which needs to be remain finite. Therefore, we define  $a/b\psi = \lambda(\psi)$  and eliminate  $a$  in favour of  $b$  and  $\psi$ . As before, we expand  $C_E$  in series of  $\epsilon$ . Thus,

$$\lambda(\psi) = \lambda_0 + \epsilon^2 \lambda_2 + \dots, b(\psi) = b_0 + \epsilon^2 b_2 + \dots, \quad (44)$$

$$C_E = C_{E0} + \epsilon^2 C_{E2} + \dots. \quad (45)$$

At lowest order, the system of equations (42)-(43) reduces to

$$\frac{d}{d\psi} \left( \frac{b_0 \lambda_0}{\psi} \right) = -2(\psi - C_{E0}) \frac{\lambda_0}{\psi} F(\psi), \quad (46)$$

$$\frac{db_0}{d\psi} = -2F(\psi), \quad (47)$$

which has the solutions

$$b_0 = -\psi^2 \left( \alpha_1 + \frac{2}{3} \alpha_2 k \psi \right) \quad (48)$$

$$\begin{aligned} \lambda_0 &= \frac{6}{7\alpha_2 k} \psi \\ &+ \frac{6}{5\alpha_2 k} \left( \frac{\alpha_1}{\alpha_2 k} - \psi \right) \left( C_{E0} + \frac{6\alpha_1}{7\alpha_2 k} \right) \psi^{-2}. \end{aligned} \quad (49)$$

In order to avoid that  $\lambda_0$  becomes singular at  $\psi = 0$  we require that  $C_{E0} = -6\alpha_1/7\alpha_2 k$ . Then,  $\lambda_0 = 6/7\alpha_2 k$  and

$$a_0 = -\frac{6}{7\alpha_2 k} \psi \left( \alpha_1 + \frac{2}{3} \alpha_2 k \psi \right). \quad (50)$$

Therefore, the phase space polynomial describing the solitary nonlinear wave regimes is approximately equal to

$$\left( \frac{d\psi}{ds} \right)^2 - \frac{6}{7\alpha_2 k} \psi \left( \alpha_1 + \frac{2\alpha_2 k}{3} \psi \right) \left( \frac{d\psi}{ds} \right) - \psi^2 \left( \alpha_1 + \frac{2\alpha_2 k}{3} \psi \right) = 0. \quad (51)$$

It shows that the auto-solitary curve has  $d\psi/ds$  become zero if  $b_0 = 0$ , which is for  $\psi = 0$  and  $\psi = -3/2\alpha_2 k$ .

The calculation of the second order solutions of Eqs. (42)-(43) are given in Appendix B. It shows that for the auto-solitary type solution the normalised envelope speed,  $C_E$ , is of the form

$$C_E = -\frac{6}{7k\alpha_2} \left[ \alpha_1 + \frac{3}{7 \cdot 49k^2} \right] + \dots. \quad (52)$$

Equation (52) represents the lower (upper) limit value of  $C_E$  for a limit cycle solution with  $\alpha_1 = 1$  and  $\alpha_2 = -1$  ( $\alpha_2 = 1$ ).

Figure 15 shows an example of the dimensionless limit cycle phase (period or wavelength) as a function of  $C_E$ . It shows

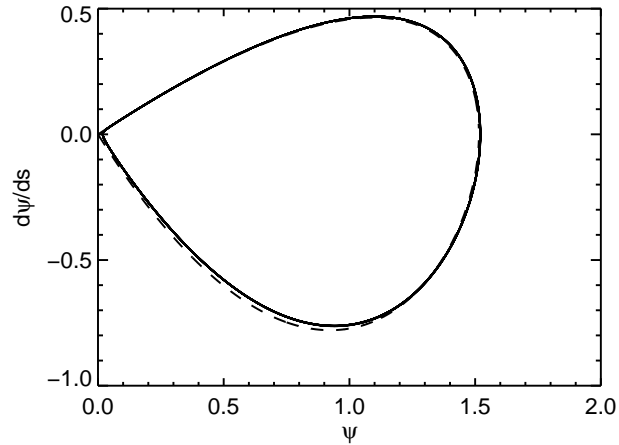


figure. 14: Normalised phase diagram  $\psi - d\psi/ds$  for  $(\alpha_1, \alpha_2) = (1, -1)$  with  $k = 1.0$  and  $C_E = 0.87$ . The solid and dashed curves are the numerical and analytical solutions, respectively

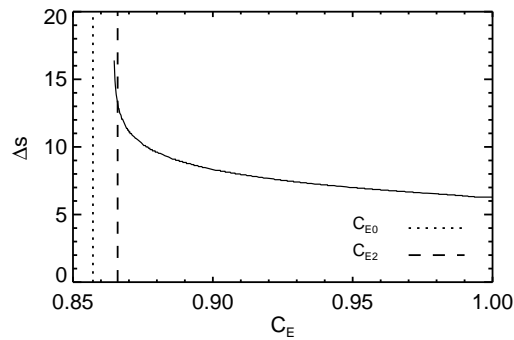


figure. 15: The dimensionless limit cycle phase,  $\Delta s$ , calculated numerically from Eq. (24), is plotted as a function of envelope speed  $C_E$  for  $(\alpha_1, \alpha_2) = (1, -1)$  and  $k=1$ .

that as  $C_E$  decreases, the oscillation period (and wavelength) grows and becomes infinite at the value of  $C_E$  given by Eq. (52). The solution is then a pulse defining a solitary wave. Therefore, using Eqs. (20), (22) and (52),  $C + ((\chi|\mu_1|)^{1/2}/\epsilon) C_E = C - (6\mu_1/7\mu_2)(1 + 3\mu_1\epsilon/313\mu_2^2\chi)$  represents the phase speed of the auto-solitary wave, which is solely dependent on plasma characteristics. To leading order, the correction to the magnetoacoustic speed depends only on the plasma activity and not on thermal diffusion. The method for generating the limit cycle without finding the long-time solution of the dynamical equation is powerful and as far as the authors are aware novel.

## VI. DISCUSSION

We have shown that in a thermally active plasma, where the heating and cooling depends quadratically on thermodynamic quantities, self-organizing magnetoacoustic waves, so-called autowaves, can exist. For the one-dimensional propagation model studied here, there is no qualitative difference between the nonlinear dynamics of the slow and fast magnetoacoustic waves. However, quantitatively, the values of the coefficients of the nonlinear evolutionary equation differs between the two types of waves. For propagation parallel to the magnetic field in a low plasma- $\beta$  plasma, only the slow wave feels the plasma activity. The fast wave degenerates into an incompressible Alfvén wave. In this limit the coefficients of activity are zero.

The sign of the linear and quadratic terms of the heating-cooling function determines the possible stationary solutions. We have shown that for a negative linear profile, i.e. active plasma, waves are amplified [3] to a stationary sawtooth-like nonlinear oscillation propagating at the magnetoacoustic speed with an amplitude independent of the initial amplitude determined by the plasma properties and the wavelength. Autowaves can only exist for a wavelength above a threshold determined by the ratio of linear activity and thermal diffusion. The extension of the heating-cooling function to include a quadratic term shows that the autowave no longer exist as the solution grows or decays in amplitude depending on the sign of the quadratic term. If the quadratic term is small compared with the linear term, then the autowave solution can maintain itself for several oscillation periods. We have named such type of solutions quasi-autowaves.

The presence of the quadratic term also leads to the existence of an auto-solitary wave, i.e. a pulse-like solution whose amplitude and propagation is completely determined by the plasma properties. Unlike the above mentioned autowaves, the solitary waves require a propagation speed that departs from the magnetoacoustic speed. From studying the phase plane dynamics, we concluded that the interplay between an unstable node and centre leads to the formation of a limit cycle. A limit cycle is defined by the preference of phase-space trajectories to evolve to a solution with a constant amplitude in phase space. A novel perturbation method has been developed to describe the highly asymmetric nature of the limit cycle as we approach a homoclinic bifurcation point in phase-space. By assuming a quadratic polynomial for the soliton shape in phase-space, the phase-space dynamics of the solitary wave and the propagation speed have been characterized.

For typical solar coronal conditions with a temperature and number density of 0.5 MK and  $5 \cdot 10^{14} \text{ m}^{-3}$ , respectively, and using the radiative loss function by [22], we find a minimum wavelength for autowaves to exist to be equal to 70 Mm. This wavelength is of the same order as the typical coronal loop length. Therefore, autowaves are expected to exist as global loop oscillations. However, the role of heating and structuring have been neglected in this simple estimate. The theory of auto-solitary waves may be applicable to ultra long-period oscillation seen in solar prominences [23] as such nonlinear solutions provide the possibility for long oscillation periods. It may also be linked to the oscillatory evolution to prominence eruption [24]. This will

be the focus of a future publication. Moreover, the steep periodic gradients in the wave profile are accompanied by periodic spikes of the current density. This can lead to the periodic onset of current-driven plasma micro-instabilities that can cause large quasi-periodic pulsations in magnetic energy releases (see [25, 26]).

### Acknowledgments

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### Appendix A: Derivation of nonlinear evolution equation

The details of the derivation of the nonlinear magnetoacoustic wave equation from the non-ideal MHD equations are given here. The perturbed version of Eq's. (6)-(5) are split into linear and quadratically nonlinear terms. Higher order nonlinear terms have been neglected. Also, we consider perturbations that spatially only depend on  $z$ . From the induction equation (4), we see that this implies that the  $z$ -component of the magnetic field perturbation is zero. The remaining equations become

$$\frac{\partial p}{\partial t} - C_S^2 \frac{\partial \rho}{\partial t} = M + N_1, \quad (\text{A1})$$

$$\frac{\partial B_x}{\partial t} + \frac{\partial}{\partial z} (B_{0x} V_z - B_{0z} V_x) = N_2, \quad (\text{A2})$$

$$\frac{\partial B_y}{\partial t} - \frac{\partial}{\partial z} (B_{0z} V_y) = N_3, \quad (\text{A3})$$

$$\rho_0 \frac{\partial V_x}{\partial t} - \frac{B_{0z}}{\mu_0} \frac{\partial B_x}{\partial z} = N_4, \quad (\text{A4})$$

$$\rho_0 \frac{\partial V_y}{\partial t} - \frac{B_{0z}}{\mu_0} \frac{\partial B_y}{\partial z} = N_5, \quad (\text{A5})$$

$$\rho_0 \frac{\partial V_z}{\partial t} + \frac{\partial p}{\partial z} + \frac{B_{0x}}{\mu_0} \frac{\partial B_x}{\partial z} = N_6, \quad (\text{A6})$$

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial V_z}{\partial z} = N_7. \quad (\text{A7})$$

where the non-adiabatic and nonlinear terms are gathered on the right-hand sides into the quantities  $N_j$  and  $M$ , given by

$$M = (\gamma-1) \left[ \rho \frac{\partial \mathcal{L}}{\partial \rho} + p \frac{\partial \mathcal{L}}{\partial p} + \frac{1}{2} \rho^2 \frac{\partial^2 \mathcal{L}}{\partial \rho^2} + \rho p \frac{\partial^2 \mathcal{L}}{\partial \rho \partial p} + \frac{1}{2} p^2 \frac{\partial^2 \mathcal{L}}{\partial p^2} + \kappa_0 \frac{\partial^2 T}{\partial z^2} \right], \quad (\text{A8})$$

$$N_1 = -V_z \frac{\partial p}{\partial z} - \frac{\rho}{\rho_0} \frac{\partial p}{\partial t} + C_S^2 V_z \frac{\partial \rho}{\partial z} + C_S^2 \frac{p}{\rho_0} \frac{\partial \rho}{\partial t}, \quad (\text{A9})$$

$$N_2 = -\frac{\partial}{\partial z} (V_z B_x), \quad (\text{A10})$$

$$N_3 = -\frac{\partial}{\partial z} (V_z B_y), \quad (\text{A11})$$

$$N_4 = -\rho \frac{\partial V_x}{\partial t} - \rho_0 V_z \frac{\partial V_x}{\partial z}, \quad (\text{A12})$$

$$N_5 = -\rho \frac{\partial V_y}{\partial t} - \rho_0 V_z \frac{\partial V_y}{\partial z}, \quad (\text{A13})$$

$$N_6 = -\rho \frac{\partial V_z}{\partial t} - \rho_0 V_z \frac{\partial V_z}{\partial z} - \frac{\partial}{\partial z} \left( \frac{B_x^2 + B_y^2}{2\mu_0} \right), \quad (\text{A14})$$

$$N_7 = -\frac{\partial}{\partial z} (\rho V_z). \quad (\text{A15})$$

$\kappa_0$  is the thermal conduction coefficient as a function of the equilibrium quantities. The terms in  $M$  are all considered to be of the same order as the nonlinear terms, and thus small. Therefore the perturbed thermodynamic quantities, using Eq. (A7), are equal to

$$\frac{\rho}{\rho_0} = -\int \frac{\partial V_z}{\partial z} dt', \quad \frac{p}{p_0} = \gamma \frac{\rho}{\rho_0}, \quad \frac{T}{T_0} = (\gamma-1) \frac{\rho}{\rho_0}. \quad (\text{A16})$$

Thus,  $M$  is rewritten as

$$M = 2\rho_0 C_S^2 \left[ L_1 + L_2 \frac{\rho}{\rho_0} + K \frac{\partial^2}{\partial z^2} \right] \frac{\rho}{\rho_0}, \quad (\text{A17})$$

where

$$L_1 = \frac{(\gamma-1)}{2C_S^2} \left( \frac{\partial}{\partial \rho} + C_S^2 \frac{\partial}{\partial p} \right) \mathcal{L}, \quad (\text{A18})$$

$$L_2 = \frac{(\gamma-1)\rho_0}{4C_S^2} \left( \frac{\partial}{\partial \rho} + C_S^2 \frac{\partial}{\partial p} \right)^2 \mathcal{L}, \quad (\text{A19})$$

$$K = \frac{(\gamma-1)^2 \kappa_0}{2\gamma \mathcal{R} \rho_0}, \quad (\text{A20})$$

$$N = \frac{1}{\rho_0} \mathcal{D}_A \left[ \frac{\partial N_6}{\partial t} - \frac{\partial}{\partial z} (C_S^2 N_7 + N_1) \right] - \frac{B_{0x}}{\rho_0 \mu_0} \frac{\partial^2}{\partial z \partial t} \left( \frac{\partial N_2}{\partial t} + \frac{B_{0z}}{\rho_0} \frac{\partial N_4}{\partial z} \right). \quad (\text{A21})$$

Equations (A3) and (A5) can be combined to yield an equation for components of the velocity and magnetic perturbations in the ignorable direction  $y$ , which is the nonlinear wave equation describing the Alfvén wave:

$$\mathcal{D}_A V_y = \frac{1}{\rho_0} \left[ \frac{\partial N_5}{\partial t} + \frac{B_{0z}}{\mu_0} \frac{\partial N_3}{\partial z} \right], \quad (\text{A22})$$

where  $\mathcal{D}_A$  is the Alfvén wave operator defined as

$$\mathcal{D}_A = \frac{\partial^2}{\partial t^2} - C_{Az}^2 \frac{\partial^2}{\partial z^2}. \quad (\text{A23})$$

Combining Eqs. (A2) and (A4) allow to write  $B_x$  in terms of  $V_z$ :

$$\mathcal{D}_A B_x + B_{0x} \frac{\partial^2 V_z}{\partial t \partial z} = \frac{\partial N_2}{\partial t} + \frac{B_{0z}}{\rho_0} \frac{\partial N_4}{\partial z}. \quad (\text{A24})$$

The linear part of this wave equation does not contain quantities associated with the magnetoacoustic waves (velocity and magnetic field perturbations in the  $x$ - $z$  plane or thermodynamic perturbations). Also, the nonlinear terms on the right-hand side of this equation depend on the product of quantities describing Alfvén ( $V_y$ ,  $B_y$ ) and magnetoacoustic waves (all other quantities). This implies that if an Alfvén wave has not been excited initially, it will not be generated in this model through linear or nonlinear coupling to the magnetoacoustic waves. Therefore, the nonlinear evolution of the magnetoacoustic waves can be studied separately from the Alfvén wave and we take  $V_y=B_y=0$ .

By combining Eqs. (A1), (A6) and (A7) it is found that

$$\begin{aligned} & \left[ \frac{\partial^2}{\partial t^2} - C_S^2 \frac{\partial^2}{\partial z^2} \right] V_z + \frac{B_{0x}}{\rho_0 \mu_0} \frac{\partial^2 B_x}{\partial t \partial z} \\ &= -\frac{1}{\rho_0} \left[ \frac{\partial M}{\partial z} + \frac{\partial N_1}{\partial z} - \frac{\partial N_6}{\partial t} + C_S^2 \frac{\partial N_7}{\partial z} \right], \end{aligned} \quad (\text{A25})$$

The nonlinear magnetoacoustic wave equation is found by eliminating  $B_x$  from Eqs. (A24) and (A26), and using Eqs. (A16)-(A17):

$$\begin{aligned} & \left[ \frac{\partial^4}{\partial t^4} - (C_A^2 + C_S^2) \frac{\partial^4}{\partial t^2 \partial z^2} + C_{Az}^2 C_S^2 \frac{\partial^4}{\partial z^4} \right] V_z \\ &= 2C_S^2 \mathcal{D}_A \left[ L_1 + K \frac{\partial^2}{\partial z^2} \right] \int \frac{\partial^2 V_z}{\partial z^2} dt' \\ &- 2C_S^2 \mathcal{D}_A L_2 \frac{\partial}{\partial z} \left[ \int \frac{\partial V_z}{\partial z} dt' \right]^2 + N, \end{aligned} \quad (\text{A26})$$

where

$$\begin{aligned} N &= \frac{1}{\rho_0} \mathcal{D}_A \left[ \frac{\partial N_6}{\partial t} - \frac{\partial}{\partial z} (C_S^2 N_7 + N_1) \right] \\ &- \frac{B_{0x}}{\rho_0 \mu_0} \frac{\partial^2}{\partial z \partial t} \left( \frac{\partial N_2}{\partial t} + \frac{B_{0z}}{\rho_0} \frac{\partial N_4}{\partial z} \right) \end{aligned} \quad (\text{A27})$$

The nonlinear terms contained in  $N$  only depend on perturbation quantities associated with the magnetoacoustic waves.

### Appendix B: Perturbation expansion

At second order, the system of equations (42)-(43) reduces to

$$\frac{da_2}{d\psi} = 2C_{E2} + \frac{\lambda_2}{\psi} F(\psi), \quad (\text{B1})$$

$$\frac{db_2}{d\psi} = a_0 \left[ \frac{\lambda_0}{\psi} F(\psi) - (\psi - C_{E0}) \right], \quad (\text{B2})$$

where

$$a_2 = \frac{\lambda_0 b_2}{\psi} + \frac{\lambda_2 b_0}{\psi}, \quad (\text{B3})$$

Substituting  $\lambda_0$  and  $C_{E0}$ , Eq. (B2) becomes

$$\frac{db_2}{d\psi} = -\frac{\lambda_0}{7} b_0, \quad (\text{B4})$$

from which we find  $b_2$  to be

$$b_2 = \frac{\lambda_0}{6 \cdot 7} \psi^3 [2\alpha_1 + \alpha_2 k \psi]. \quad (\text{B5})$$

Using Eq. (47), Eq. (B2) reduces to

$$\frac{d}{d\psi} \left( \frac{b_0 \sqrt{|b_0|} \lambda_2}{\psi} \right) = \left[ 2C_2 - \lambda_0 \frac{d}{d\psi} \left( \frac{b_2}{\psi} \right) \right] \sqrt{|b_0|}. \quad (\text{B6})$$

Using a change of variable to  $\psi = (z - \alpha_1) \left( \frac{3}{2\alpha_2 k} \right)$ , Eq. (B6) is readily integrated:

$$\lambda_2 = -\frac{\lambda_0^2 \alpha_2}{4 \cdot 7k} \frac{\left[ z^3 - \frac{19\alpha_1}{7} z^2 + (11-d) \frac{z}{5} + (d-1) \frac{\alpha_1}{3} \right]}{(z - \alpha_1)^2} \quad (\text{B7})$$

Where  $d = 14 \cdot 8\alpha_2 C_{E2} k / \lambda_0^2$ . For  $\psi = 0$ , which corresponds to  $z = \alpha_1$ , the denominator becomes zero. In order for  $\lambda_2$  to remain finite, the numerator must vanish as well for  $z = \alpha_1$ . This forces the parameter  $d$  to be equal to  $d = -8/7$ . Therefore, we find a condition for  $C_{E2}$ :

$$C_{E2} = -\frac{18}{7^4 \alpha_2 k^3} \quad (\text{B8})$$

Equation (B9) then becomes;

$$\lambda_2(\psi) = \frac{\lambda_0^2 \alpha_2}{6 \cdot 7k} [3\alpha_1 + \alpha_2 k \psi]. \quad (\text{B9})$$

Note that while  $\lambda$  is a constant to leading order, its second order correction linearly depends on  $\psi$ , i.e. Using Eqs. (B3) and (B9),  $a_2$  is found to be

$$a_2 = \frac{\lambda_0^2 \psi}{42} \left[ -\frac{3\alpha_2}{k} - \alpha_1 \psi + \frac{k\alpha_2}{3} \psi^2 \right]. \quad (\text{B10})$$

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- [1] R. V. Jensen, D. E. Post, W. H. Grasberger, C. B. Tarter, and W. A. Lokke, *Nucl. Fusion* **17**, 1187 (1977), ISSN 0029-5515.
- [2] J. C. Raymond and B. W. Smith, *Astrophys. J. Suppl.* **35**, 419 (1977).
- [3] G. B. Field, *Astrophys. J.* **142**, 531 (1965).
- [4] G. W. Pneuman, *Astrophys. J.* **177**, 793 (1972).
- [5] H. Koyama and S.-i. Inutsuka, *Astrophys. J.* **564**, L97 (2002).
- [6] W. M. Stacey, *Fusion Sci. Tech.* **52**, 29 (2007).
- [7] E. N. Parker, *Astrophys. J.* **117**, 431 (1953).
- [8] M. Kruskal and M. Schwarzschild, in *Magnetohydrodynamic Stability and Thermonuclear Containment*, (New York: Academic), (1965).
- [9] R. B. Dahlburg, C. R. Devore, J. M. Picone, J. T. Mariska, and J. T. Karpen, *Astrophys. J.* **315**, 385 (1987).
- [10] J. T. Karpen, S. K. Antiochos, J. M. Picone, and R. B. Dahlburg, *Astrophys. J.* **338**, 493 (1989).
- [11] F. Reale, S. Serio, and G. Peres, *Astrophys. J.* **433**, 811 (1994).
- [12] D. A. N. Müller, V. H. Hansteen, and H. Peter, *A&A* **411**, 605 (2003).
- [13] V. M. Nakariakov and B. Roberts, *Phys.Lett.A*, **254**, 314 (1999).
- [14] H. Haken, *Synergetics*, (Springer-Verlag, Berlin), (1983).
- [15] V. M. Nakariakov, C. A. Mendoza-Briceño, and M. H. Ibáñez S., *Astrophys. J.* **528**, 767 (2000), arXiv:astro-ph/9909318.
- [16] A. Kelly and V. M. Nakariakov, in *SOHO 13 Waves, Oscillations and Small-Scale Transients Events in the Solar Atmosphere: Joint View from SOHO and TRACE*, edited by H. Lacoste (2004), vol. 547 of *ESA Special Publication*, p. 483.
- [17] S. I. Braginskii, *Rev. Plasma Phys.*, ed M. A. Leontovich (New York: Consultants Bureau) (1965).
- [18] W. M. Stacey, *Fusion Plasma Physics*, (Wiley-VCH Verlag GmbH & Co. KGaA, Weinheim), (2005).
- [19] B. Van der Pol and J. Van der Mark, *Nature* **120**, 363 (1927).
- [20] H. T. Davies, *Introduction to nonlinear differential and integral equations*, (Dover, Washington), (1962).
- [21] J. D. Hoffman, *Numerical Methods for Engineers and Scientists*, (McGraw-Hill Inc, New York), (1992).
- [22] R. Rosner, W. H. Tucker, and G. S. Vaiana, *Astrophys. J.* **220**, 643 (1978).
- [23] C. Foullon, E. Verwichte, and V. M. Nakariakov, *A&A* **427**, L5 (2004).
- [24] C. Foullon, E. Verwichte, and V. M. Nakariakov, *Astrophys. J.* **700**, 1658 (2009).
- [25] V. M. Nakariakov, C. Foullon, E. Verwichte, and N. P. Young, *A&A* **452**, 343 (2000).
- [26] C. Foullon, E. Verwichte, V. M. Nakariakov, and L. Fletcher, *A&A* **440**, L59 (2005).