Fair Division of Indivisible Goods

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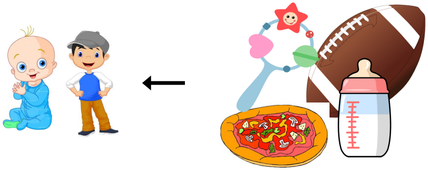
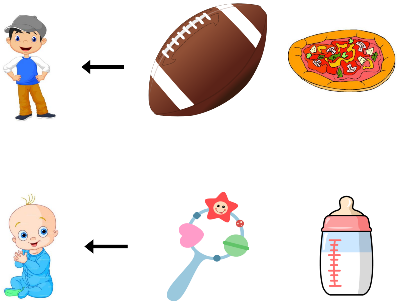
Fair Division

Divide *items* among agents *fairly*



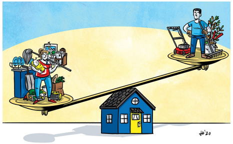


Applications

Vaccine distributions

Divorce settlements

Air traffic management



Household chores

Setup (Discrete Fair Division)

Given:

- Set $[n]$ of n agents.
- Set M of m indivisible goods.
- Additive valuations $v_i \colon 2^M \to \mathbb{R}_{\geq 0}$ for every agent i.

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Find: Partition $X = \langle X_1, X_2, \ldots, X_n \rangle$ of M, which is *fair*.

Fairness Notions

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Is it always possible to be fair? (notion being Envy-Freeness)

Answer

NO! Consider two agents having positive valuation towards a single good.

Relaxation: Envy-Freeness up to Any Good (EFX)

(CKMPSW’16)

X is EFX iff for all i, i', $v_i(X_i) \geq v_i(X_{i'} \setminus \{g\})$ for *every* $g \in X_{i'}$.

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35	a_1	25	10	10
10	a_2	20	10	10

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$v_2(\{\text{Shoes}\}) < v_2(\{\text{Laptop}\})$

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An EFX allocation

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Question

Is it always possible to be fair? (notion being EFX)

Answer– We do not know yet!

“Fair division’s biggest problem” – Procaccia (CACM’20)

“highly non-trivial” (even for 3 agents) – Plaut and Roughgarden (SODA’18)

State of the Art (EFX)

$n = 2$	$n = 3$	$n > 3$
Exists (PR'18)	Exists (C**G**.M'20)	**Open**

Relaxations

1. EFX with *charity* (CKMS'20, BCFF'21, M'21)
    - EFX with at most $n - 2$ unallocated goods
2. *Approximate*-EFX (PR'18, ANM'20). $v_i(X_i) \geq \alpha v_i(X_j \setminus g) \; \forall g \in X_j$ for $\alpha \in [0, 1]$
    - 0.618-approximate EFX
3. Approximate-EFX with *charity* (C**G**.MMM'21)
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Two Agents: Divide and Choose

- Agent 1 finds a partition (Y, Y') of all goods such that

$$v_1(Y') \geq v_1(Y) \geq v_1(Y' \setminus g), \forall g \in Y'$$

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Theorem

EFX exists when $n = 2$

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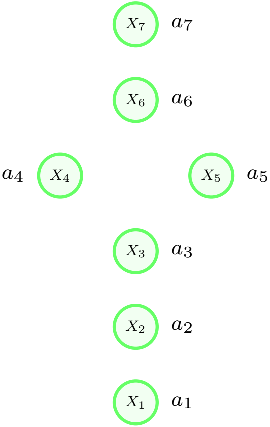
Concepts: Envy Graph E_X

- Vertices correspond to agents $[n]$.
- $(i, j) \in E_X$ iff $v_i(X_i) < v_i(X_j)$.



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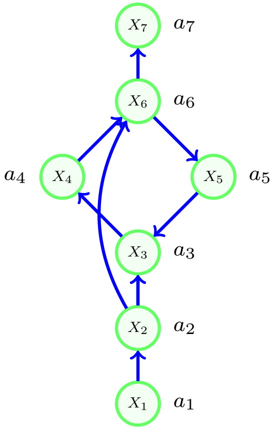

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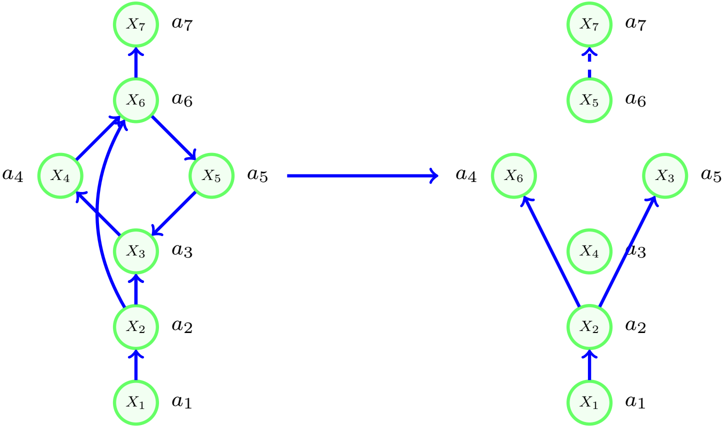

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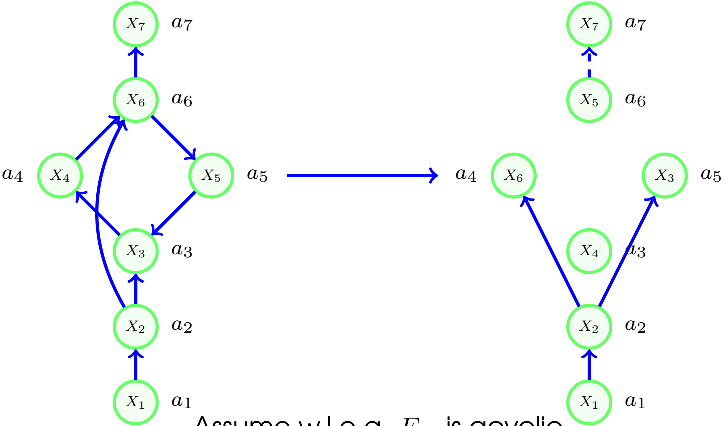
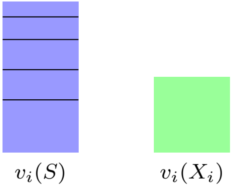


Assume w.l.o.g. E_X is acyclic

Concepts: Most Envious Agents $A_X(S)$

Given: $X = \langle X_1, \dots, X_n \rangle$ and $S \subseteq M$ and agent i.



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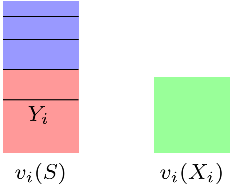

Let Y_i be the smallest subset of S such that $v_i(Y_i) > v_i(X_i)$.

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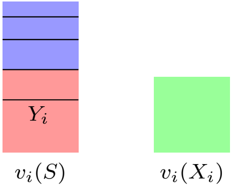


Let Y_i be the smallest subset of S such that $v_i(Y_i) > v_i(X_i)$.

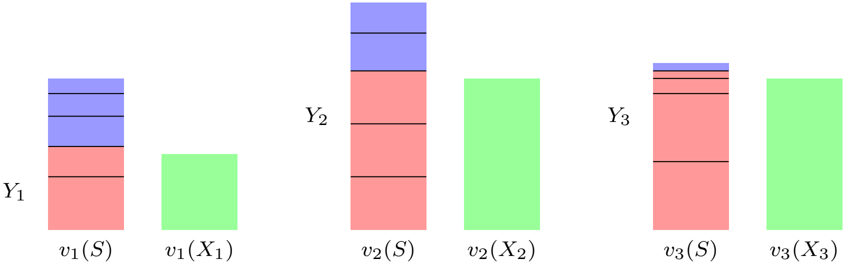
Define $\kappa_X(i, S) = |Y_i|$.

Concepts: Most Envious Agents $A_X(S)$

$A_X(S) =$ agents with minimum $\kappa_X(i, S)$.

Concepts: Most Envious Agents $A_X(S)$

$A_X(S)$ = agents with minimum $\kappa_X(i,S)$.



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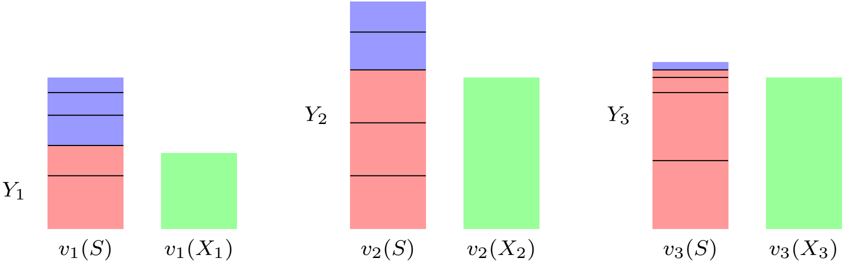
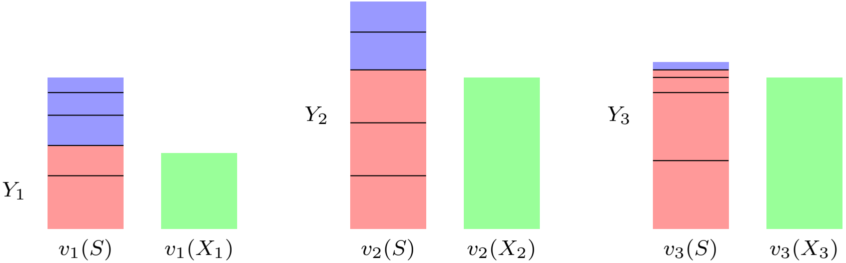


$$\boxed{A_X(S) = \{a_1\}}$$

Concepts: Most Envious Agents $A_X(S)$

$$A_X(S) = \text{agents with minimum } \kappa_X(i, S).$$



Nobody envies Y_1 up to any good and $v_1(Y_1) >_1 v_1(X_1)$!

Concepts: Champions and Champion-Cycle

Given allocation X, we say i **champions** the set S,

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Given allocation X, we say i **champions** the set S, if i is a **most envious agent** for S.

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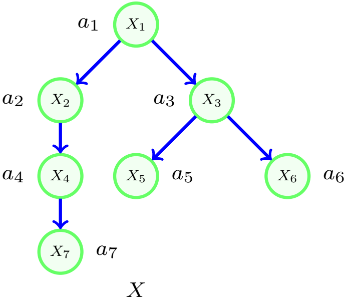
Given: a partial EFX allocation $X = \langle X_1, \ldots, X_n \rangle$

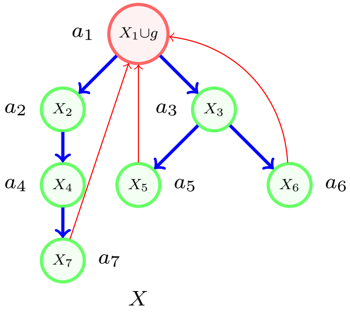
E_X has a single source



Concepts: Champions and Champion-Cycle

Given: a partial EFX allocation $X = \langle X_1, \ldots, X_n \rangle$

Pick any unallocated g



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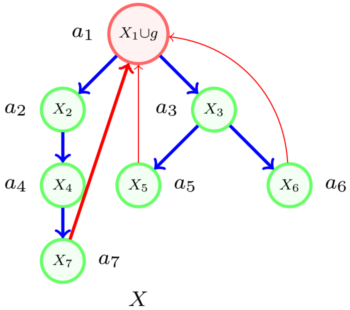

$A_X(X_1 \cup g) = \{a_7\}$

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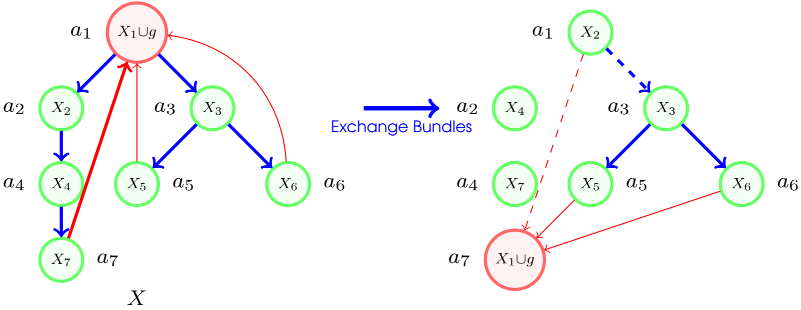

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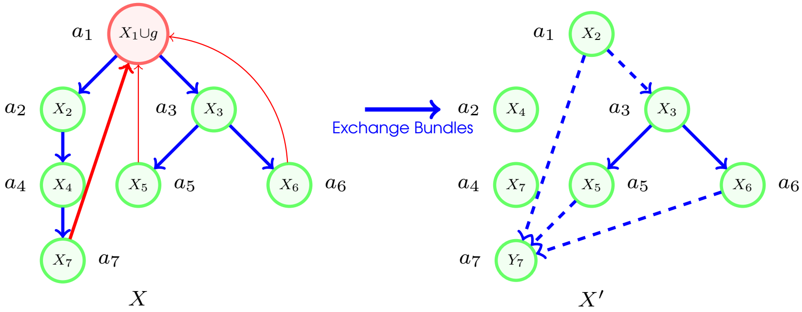

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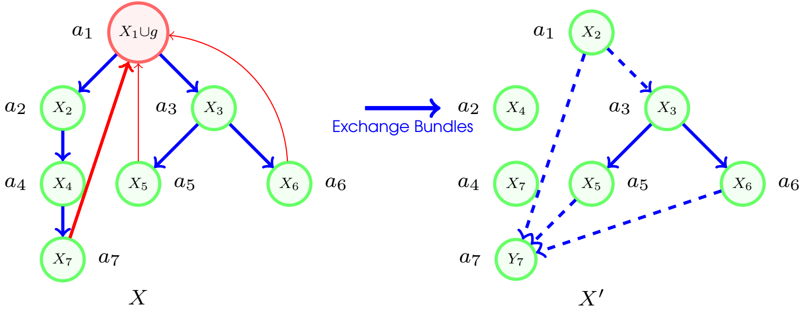
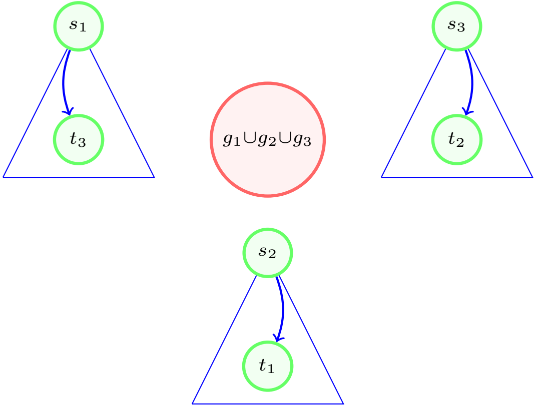


$X' >_{PD} X$ and X' is EFX

Concepts: Champions and Champion-Cycle

Agents t_1, t_2, t_3 and goods g_1, g_2, g_3 form a champion-cycle

t_i belongs to the component in E_X with s_i as source



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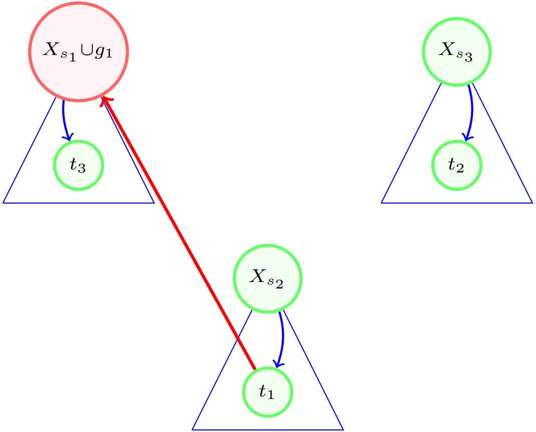
t_1 champions $X_{s_1} \cup g_1$



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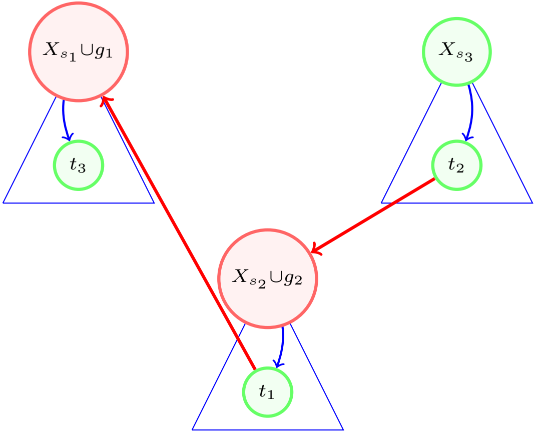

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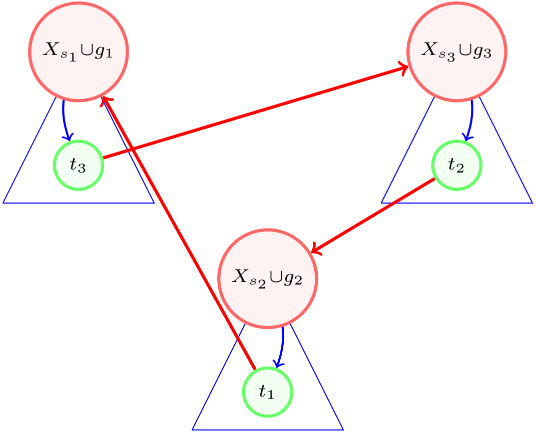
t_3 champions $X_{s_3} \cup g_3$



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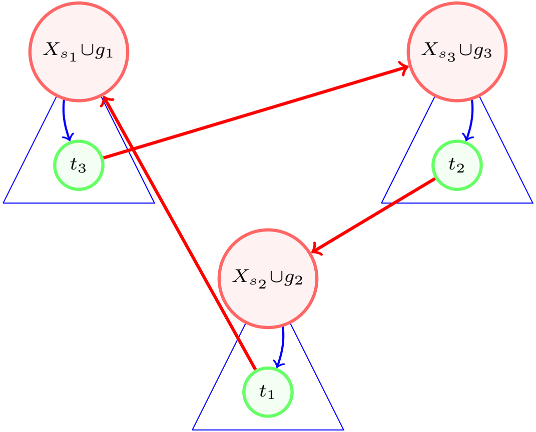

If the number of unallocated goods is at least n, then X admits champion-cycle

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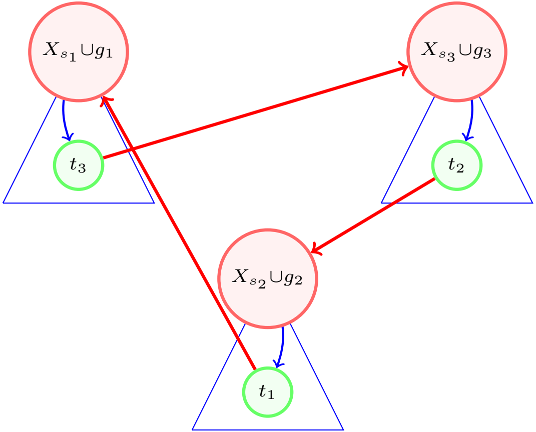

X **admits champion-cycle** $\implies$ **there exists EFX allocation** $X' >_{PD} X$

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Exists (PR'18)	Exists (C**G**.M'20)	**Open**

Relaxations

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Three Agents – First Attempt

```
Invariant: $X$ is EFX
For all $i \in [n]$ set $X_i \leftarrow \emptyset$
while there is an unallocated good $g$ do
    $X \leftarrow U(X, g)$ by some update rule $U$
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$X' = U(X, g)$. Then

- X' is EFX

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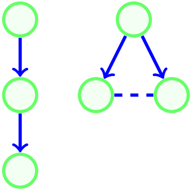
This strategy has been useful before:

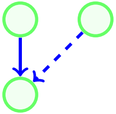
- 0.5-EFX (PR'18)
- EFX with charity (CKMS'20)

Three Agents: Sources in E_X

- One source



- Two sources



- Three sources

Case 1: E_X Has a Single Source

(CKMS'20)

Given: A partial EFX allocation X and an unallocated good g such that E_X has a single source.

Then there exists a partial EFX allocation $X' >_{PD} X$

Case 2: E_X Has Three Sources

(C**G.**M'20)

Given: A partial EFX allocation X and an unallocated good g such that E_X has three sources.

Then there exists a partial EFX allocation $X' >_{PD} X$

Case 2: E_X Has Three Sources

Sketch

$\tilde{X}_i \subseteq X_i \cup g$ of smallest size, such that $v_i(\tilde{X}_i) > v_i(X_i)$

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$\tilde{X}_i \subseteq X_i \cup g$ of smallest size, such that $v_i(\tilde{X}_i) > v_i(X_i)$

$\downarrow$

If $\langle \tilde{X}_1, X_2, X_3 \rangle$ or $\langle X_1, \tilde{X}_2, X_3 \rangle$ or $\langle X_1, X_2, \tilde{X}_3 \rangle$ is EFX, then exit

Case 2: E_X Has Three Sources

Sketch

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↓

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↓

Otherwise, there exists $X' >_{PD} X$ and $\cup_{i \in [n]} X'_i = \cup_{i \in [n]} X_i$

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$\tilde{X}_i \subseteq X_i \cup g$ of smallest size, such that $v_i(\tilde{X}_i) > v_i(X_i)$

↓

If $\langle \tilde{X}_1, X_2, X_3 \rangle$ or $\langle X_1, \tilde{X}_2, X_3 \rangle$ or $\langle X_1, X_2, \tilde{X}_3 \rangle$ is EFX, then exit

↓

Otherwise, there exists $X' >_{PD} X$ and $\cup_{i \in [n]} X'_i = \cup_{i \in [n]} X_i$

↓

Create EFX allocation Y from X' and g such that $Y >_{PD} X'$

Case 3: E_X Has Two Sources: No U Possible!

Lemma

There exists a partial EFX allocation X and an unallocated good g, such that there exists no complete EFX allocation $X' >_{PD} X$

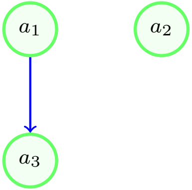
Case 3: E_X Has Two Sources: No U Possible!

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
a_1	8	2	12	2	0	17	1
a_2	5	0	9	4	10	0	3
a_3	0	0	0	0	9	10	2

Case 3: E_X Has Two Sources: No U Possible!

↓ "unallocated"

$$
\begin{array}{ccccccccc}
 & & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 \\
16 & a_1 & 8 & \multicolumn{3}{c}{\boxed{2 \qquad 12 \qquad 2}} & 0 & 17 & 1 \\
15 & a_2 & \boxed{5} & 0 & 9 & 4 & \boxed{10} & 0 & 3 \\
10 & a_3 & 0 & 0 & 0 & 0 & 9 & \boxed{10} & 2
\end{array}
$$



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In all final EFX allocations, at least one agent's valuation **strictly** decreases!

Case 3: E_X Has Two Sources: No U Possible!

$$\begin{array}{cccccccccc}
 & & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 & \\
16 & a_1 & 8 & 2 & 12 & \boxed{2} & 0 & \boxed{17} & 1 & \mathbf{19} \\
15 & a_2 & \boxed{5} & 0 & \boxed{9} & 4 & 10 & 0 & 3 & 14 \\
10 & a_3 & 0 & 0 & 0 & 0 & \boxed{9} & 10 & \boxed{2} & \mathbf{11}
\end{array}$$

a_1 and a_3 are strictly better off, while a_2 is worse off

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\end{array}$$

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For each $i \in [3]$, there is a complete EFX allocation where a_i is better off!

Fix: New Potential Function $\phi(X)$

- Relabel agents as a, b and c

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Given any partial EFX allocation X and an unallocated good g, there exists another partial EFX allocation X' such that $\phi(X') >_{lex} \phi(X)$

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Theorem
EFX exists when $n = 3$

State of the Art (EFX)

$n = 2$	$n = 3$	$n > 3$
Exists (PR'18)	Exists (C**G.**M'20)	**Open**

Relaxations

1. EFX with *charity* (CKMS'20, BCFF'21, M'21)
   - EFX with at most $n - 2$ unallocated goods
2. *Approximate*-EFX (PR'18, ANM'20). $v_i(X_i) \geq \alpha v_i(X_j \setminus g)\ \forall g \in X_j$ for $\alpha \in [0, 1]$
   - 0.618-approximate EFX
3. Approximate-EFX with *charity* (C**G.**MMM'21)
   - $(1 - \varepsilon)$-EFX with $\mathcal{O}((n/\varepsilon)^{\frac{4}{5}})$ charity for $\epsilon > 0$

EFX Allocations with *Sublinear* Charity

Almost EFX with sublinear charity $\rightarrow_{reduces}$ extremal graph theory problem.

Reduction Sketch: Goods Classification

Good g **is valuable to** i **iff** $v_i(g) > \varepsilon \cdot v_i(X_i)$**.**

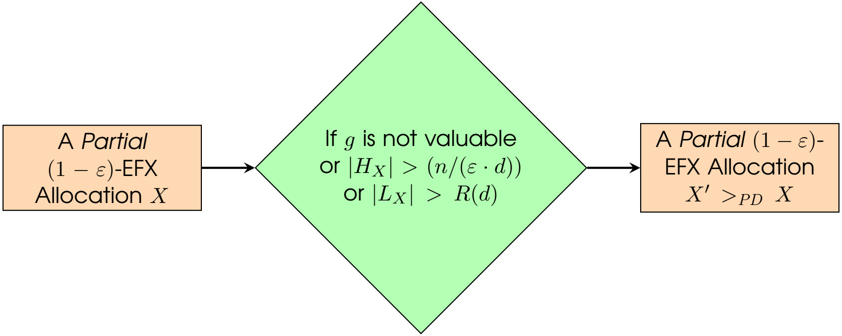
Reduction Sketch: Goods Classification

Good g is valuable to i iff $v_i(g) > \varepsilon \cdot v_i(X_i)$.

- High Demand Goods H_X.
- $g \in H_X$, iff g is valuable to at least $d+1$ agents.

- Low Demand Goods L_X.
- $g \in L_X$, iff g is valuable to at most d agents.

Reduction Sketch



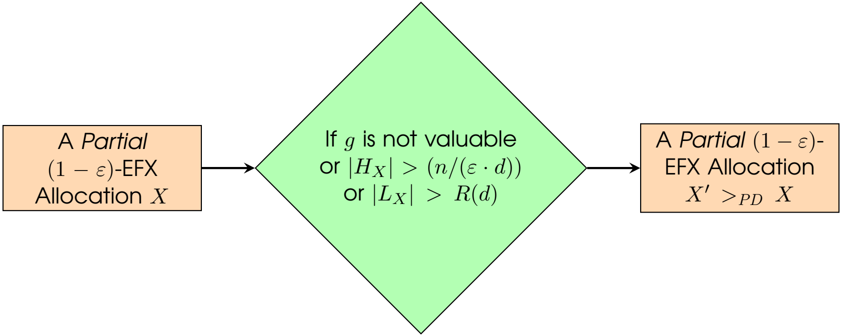
A *Partial* $(1-\varepsilon)$-EFX Allocation X

If g is not valuable or $|H_X| > (n/(\varepsilon \cdot d))$ or $|L_X| > R(d)$

A *Partial* $(1-\varepsilon)$-EFX Allocation $X' >_{PD} X$

$X' >_{PD} X$ iff $v_i(X'_i) \geq v_i(X_i)$ for all i, with at least one strict inequality.

Reduction Sketch



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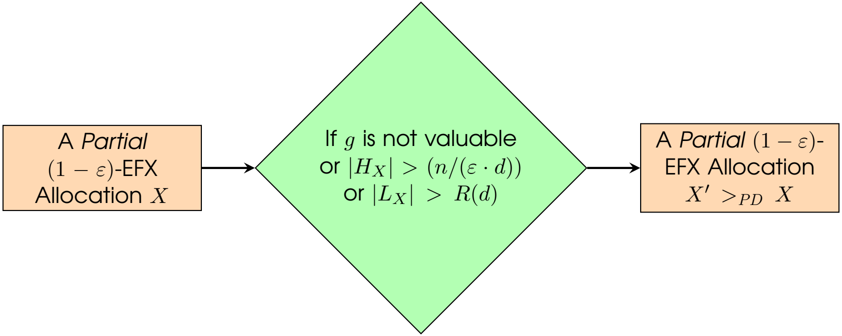
A *Partial* $(1 - \varepsilon)$-EFX Allocation $X' >_{PD} X$

Process will converge to EFX allocation where $|H_X| + |L_X| \leq n/(\varepsilon d) + R(d)$.

Reduction Sketch



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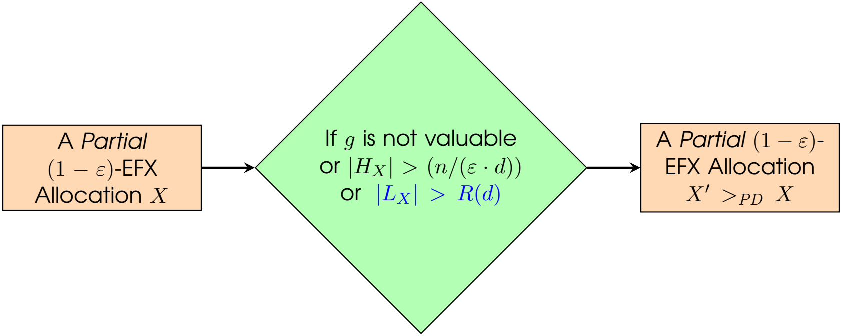

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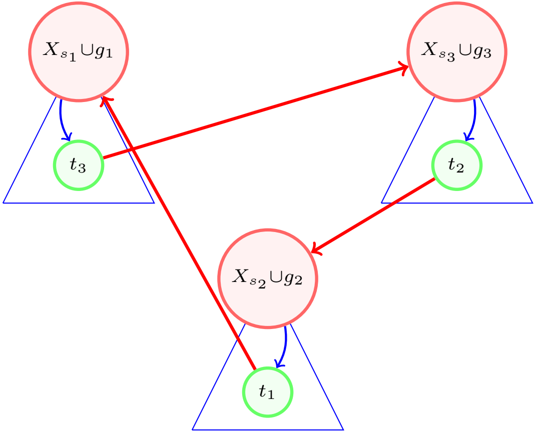
Concepts: Champions and Champion-Cycle

Agents t_1, t_2, t_3 and goods g_1, g_2, g_3 form a champion-cycle

t_i belongs to the component in E_X with s_i as source



X **admits champion-cycle** $\implies$ **there exists EFX allocation** $X' >_{PD} X$

Bounding L_X: Group Champion Graph

$$L_X = \{g_1, g_2, g_3\}$$

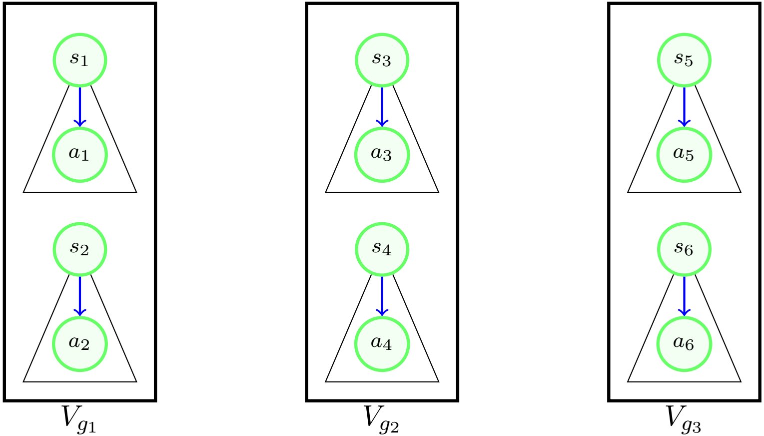
$V_{g_1} =$ components in E_X containing agents who find g_1 valuable

$V_{g_2} =$ components in E_X containing agents who find g_2 valuable

$V_{g_3} =$ components in E_X containing agents who find g_3 valuable

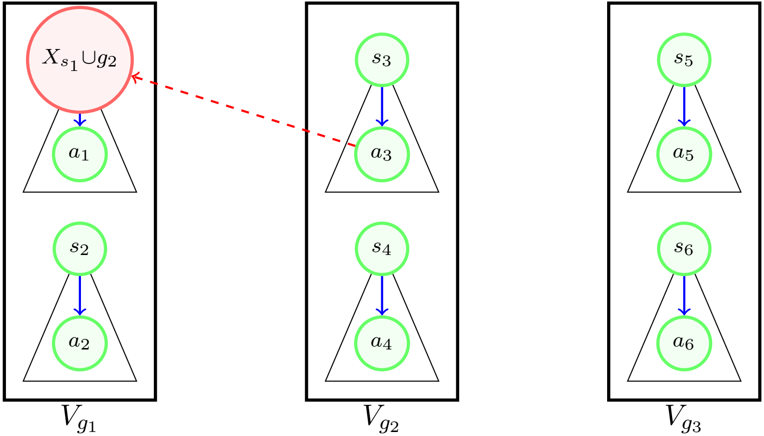
Bounding L_X: Group Champion Graph

$|V_{g_i}| \leq d$



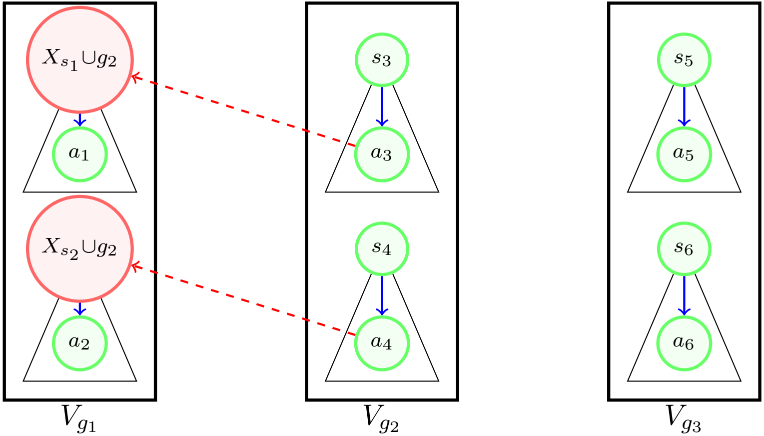
Bounding L_X: Group Champion Graph

Edges are champion edges



Bounding L_X: Group Champion Graph

The champion edges for $X_{s_i} \cup g_2$ come from V_{g_2}



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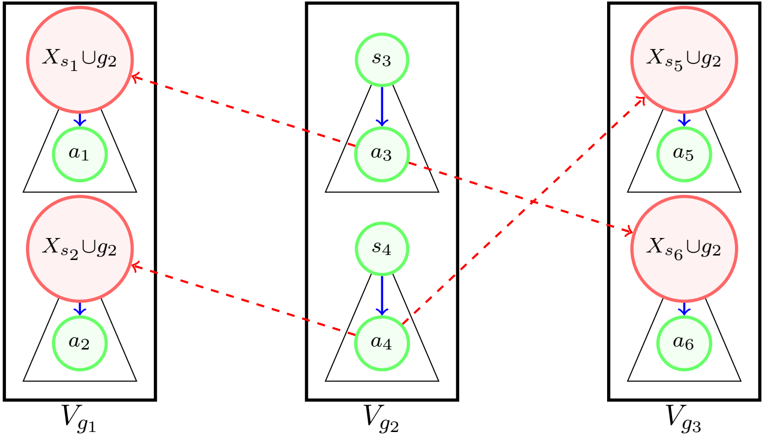
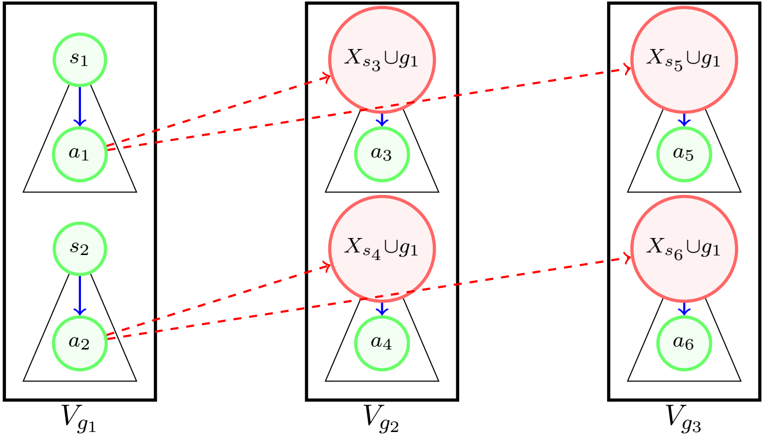


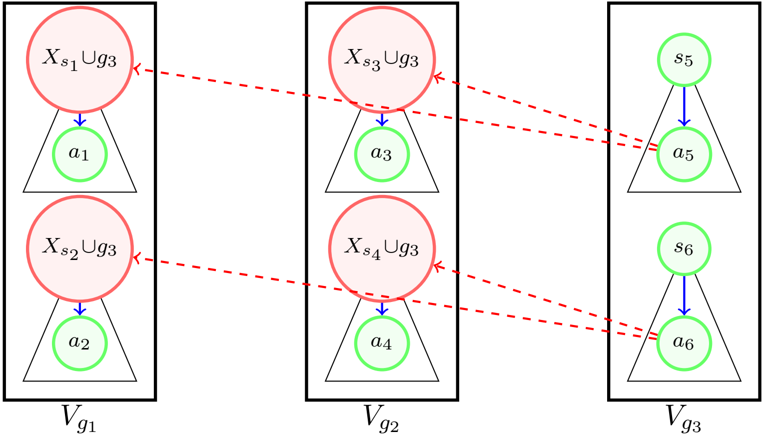
Bounding L_X: Group Champion Graph

The champion edges for $X_{s_i} \cup g_1$ come from V_{g_1}



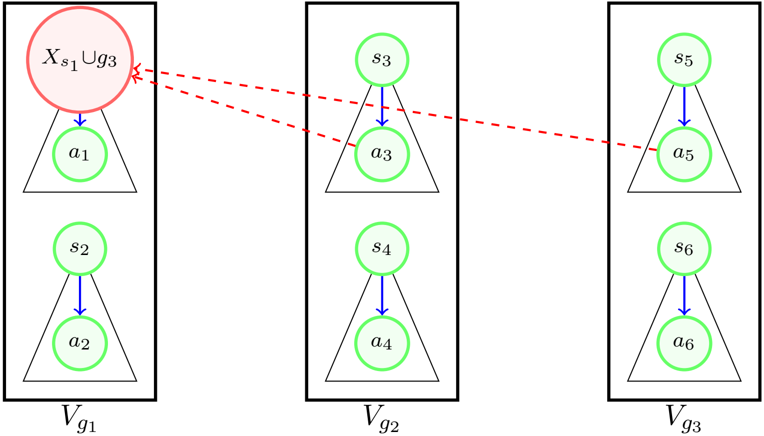
Bounding L_X: Group Champion Graph

The champion edges for $X_{s_i} \cup g_3$ come from V_{g_3}



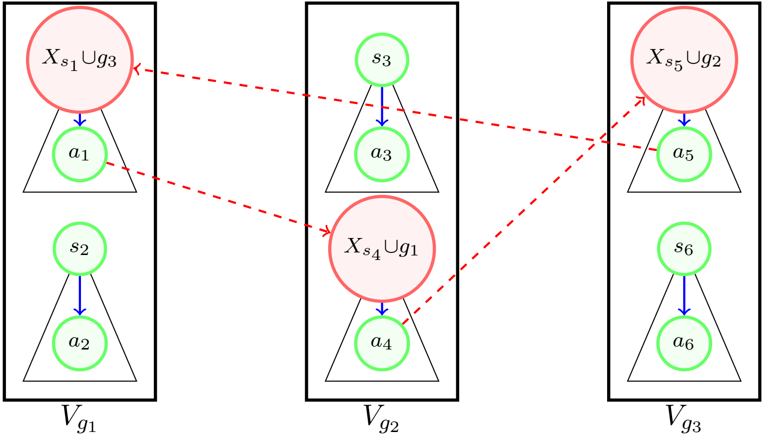
Bounding L_X: Group Champion Graph

Each vertex in V_{g_i} has incoming edge from V_{g_j}



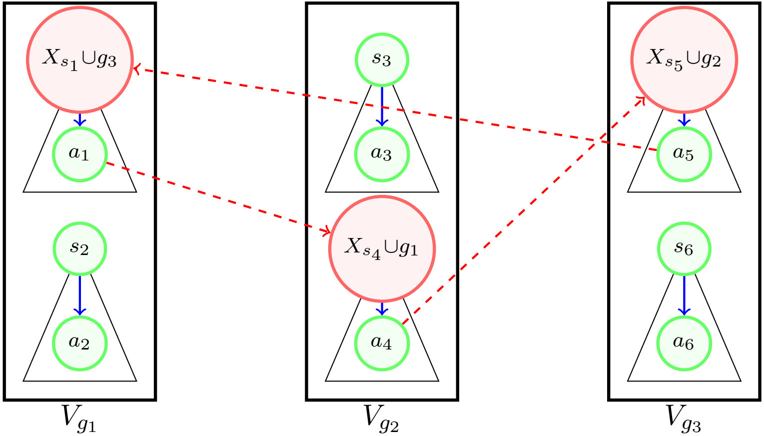
Bounding L_X: Group Champion Graph

Existence of a cycle that visits each part at most once implies existence of champion-cycle



Bounding L_X: Group Champion Graph

Question: How many parts can we have such that there is no such cycle?



Bounding L_X: Rainbow Cycle Number

Main Question

Find the largest k s.t. there is a k-partite graph $G = (\cup_{i\in[k]} V_i, E)$, where

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If $|L_X| > R(d)$, then there exists an EFX allocation $X' >_{PD} X$.

Rainbow Cycle Number

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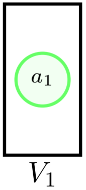

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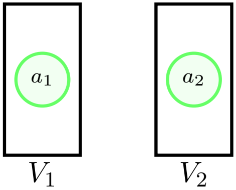

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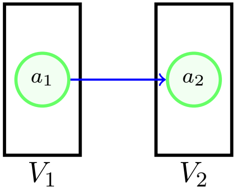

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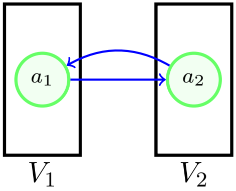

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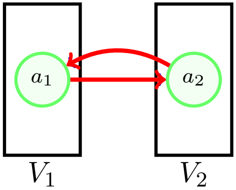

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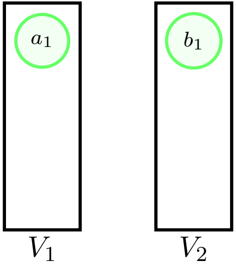

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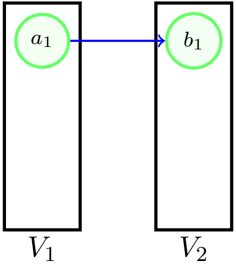

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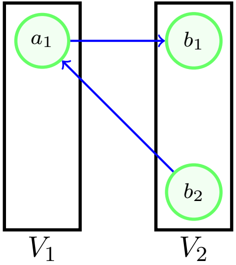

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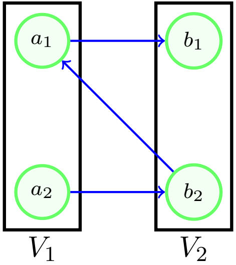

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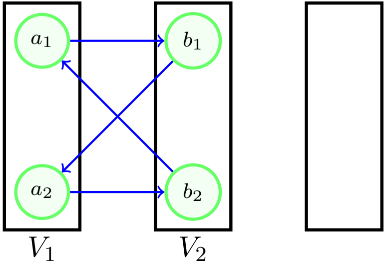

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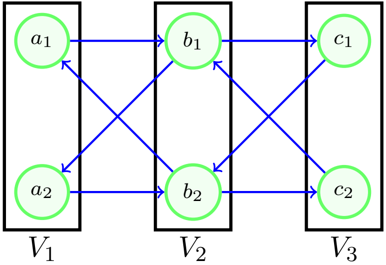

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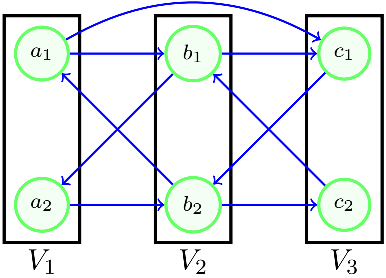

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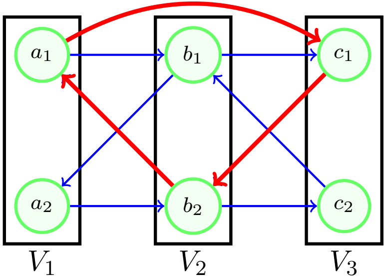

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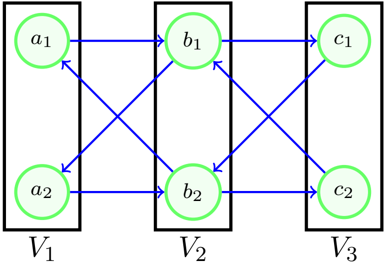

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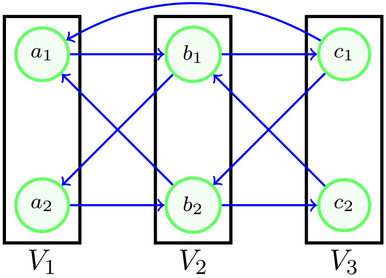

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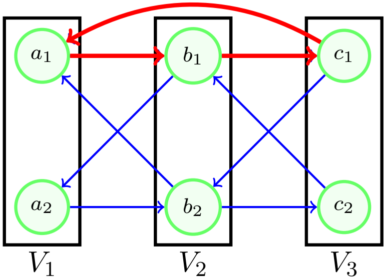

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$R(d) \in \mathcal{O}(d^4)$

- $|V_i| \leq d$ for all i,
- each vertex in V_i has an incoming edge from V_j and vice-versa, and
- there is no cycle that visits each part at most once.

Rainbow Cycle Number

$R(d) \in \mathcal{O}(d^4) \implies$ existence of $(1-\varepsilon)$-EFX with $\mathcal{O}((n/\varepsilon)^{4/5})$ charity.

- $|V_i| \leq d$ for all i,
- each vertex in V_i has an incoming edge from V_j and vice-versa, and
- there is no cycle that visits each part at most once.

Open Problems

$n = 2$	$n = 3$	$n > 3$
Exists (PR'18)	Exists (C**G**.M'20)	**Open**

Relaxations

1. EFX with *charity* (CKMS'20, BCFF'21, M'21)
    - EFX with at most $n - 2$ unallocated goods
2. *Approximate*-EFX (PR'18, ANM'20). $v_i(X_i) \geq \alpha v_i(X_j \setminus g)\ \forall g \in X_j$ for $\alpha \in [0, 1]$
    - 0.618-approximate EFX
3. Approximate-EFX with *charity* (C**G**.MMM'21)
    - $(1 - \varepsilon)$-EFX with $\mathcal{O}((n/\varepsilon)^{\frac{4}{5}})$ charity for $\epsilon > 0$

Thank you!

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