Linearly sized induced **odd** subgraphs

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A very classical stuff

Gallai's Theorem (60's): $G = (V, E)$ − graph

$\exists$ partition $V = V_1 \cup V_2$ s.t. both induced subgraphs

$G[V_1], G[V_2]$ have all degrees even

(see, e.g., CP&E of Lovász, Problem 5.17 for a proof by Pósa)

Conclusion: Can also partition $V = V_1 \cup V_2$

$G[V_1]$ – all degrees even; $G[V_2]$ − all degrees odd

[Proof: Add v to G, connect v to all of $V(G)$, get G';

apply Gallai to G' to get $V(G') = V_1' \cup V_2'$;

delete v from G']

Conclusion: $\forall G = (V, E)$ contains $V_1 \subseteq V(G), |V_1| \geq \frac{|V|}{2}$,

$G[V_1]$ has all degrees even.

Let there be light…

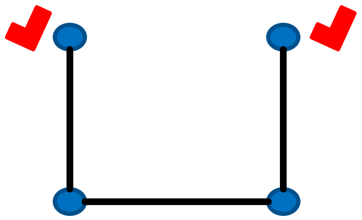
A riddle for you:

$G = (V, E)$ – graph

- each vertex $v \in V$ has a light and a button
- pressing button at v: switches the light status for v and all its neighbors
- start with all lights off

Prove: can push some buttons to get all lights on

Ex.: $G =$



Odd things are odd…

- e + e ✓
- e + o ✓

? Can we do o+o ?

No: $G = (V, E)$ – all degrees odd $\Rightarrow |V|$ – even

not every graph G has even # of vertices…

Odd things are odd… (cont.)

A side remark:

For which graph $G = (V, E)$ can we partition

$V = V_1 \cup \cdots \cup V_k$ (for some $k \geq 1$) s.t.

$\forall 1 \leq i \leq k$, $G[V_i]$ has all degrees odd?

Scott'01: $V(G)$ can be partitioned into induced odd subgraphs

$\Leftrightarrow$ every connected component of G has even order

(so called perfect forest theorem;

can require: $\forall\, V_i$ contains an induced spanning tree)

Odd things are odd… (cont.)

Also:

v is isolated in G $\Longrightarrow$ v is never a part of any odd subgraph

$\Longrightarrow$ should assume: $\delta(G) \geq 1$.

Notation:

$f_o(G) := \max\{|V_0|: V_0 \subseteq V(G), G[V_0]$ has all degrees odd$\}$

$f_o(n) := \min\{f_o(G): G = (V,E), |V| = n, \delta(G) \geq 1\}$

So a conjecture…

Conjecture

(stated in Caro'94,

"certainly a part of the graph theory folklore"):

∃ constant $c > 0$ s.t.

$$f_o(n) \geq cn, \forall n \in \mathbb{N}$$

Previous results

- Caro'94 (+Alon): $f_o(n) \geq c\sqrt{n}$;
- Scott'92: $f_o(n) \geq \frac{cn}{\log n}$;

  $G \sim G(n, \frac{1}{2})$: whp $f_o(n) = (1 + o(1))cn$ for $c = 0.7729 \ldots$

- Bounds on $f_o(G)$ thru: – max degree $\Delta(G)$;
  – independence number $\alpha(G)$;
  – chromatic number $\chi(G)$;
  – etc.

Main result

Th. 1: Every graph $G = (V, E), |V| = n, \delta(G) \geq 1,$

contains a subset $V_0 \subseteq V(G), |V_0| \geq \frac{n}{10{,}000}$

s.t. $G[V_0]$ has all degrees odd.

I.e.:

$$f_o(n) \geq \frac{n}{10{,}000}$$

– settling the conjecture.

Application – covering by odd subgraphs

Scott'01:

$t(G) \coloneqq \min\{t : \exists\, V_1, \dots, V_t$ (not necess. disjoint),

$G[V_i]$ – odd, $V(G) = \bigcup_{i=1}^{t} V_i\}$

$t(n) \coloneqq \max\{t(G) : |V(G)| = n, \delta(G) \geq 1\}$.

Scott: $c \log n \leq t(n) \leq C \log^2 n$

Covering by odd subgraphs (cont.)

Cor.: $t(n) = \Theta(\log n)$

Proof: Upper bound: apply repeatedly Th. 1 to find

$V_1, \dots, V_t$ s.t.:

- $V_i \subseteq V \setminus \bigcup_{j=1}^{i-1} V_j$;
- $G[V_i]$ odd;
- V_i covers positive % of non-isolated vertices in $G[V \setminus \bigcup_{j=1}^{i-1} V_j]$.

Stop after $O(\log n)$ steps, with V^*, $G[V^*]$ – independent set

Can cover V^* with further $O(\log n)$ sets (Scott; see later).

Covering by odd subgraphs (cont.)

Lower bound (Scott):

G := 1-subdivision of K_k ($|V(G)| = k + \binom{k}{2} = \Theta(k^2)$)

$(V_1, \dots, V_t)$ – cover by odd subgraphs

v – subdivision vertex ($d_G(v) = 2$)

u_1, u_2 – neighbors of v

$V_i \ni v \Rightarrow V_i$ contains exactly one of u_1, u_2

Conclusion: $(V_1, \dots, V_t)$ separates $[k]$

$$\Longrightarrow t = \Omega(\log k) = \Theta(\log |V(G)|). \qquad \blacksquare$$

Proof ingredients 1

Lemma 1: $f_o(G) \geq \frac{\Delta(G)}{2}$.

Proof: v ≔ vertex of max degree

$U \subseteq N_G(v), |U|$ – odd, $|U| \geq \Delta(G) - 1$

Apply Gallai to $G[U]$ to get $U = V_1 \cup V_2$,

$G[V_1]$ – even, $G[V_2]$ – odd

(⇒ $|V_2|$ – even ⇒ $|V_1|$ – odd)

Then: $G[V_1 + v], \mathrm{G}[\mathrm{V}_2]$ – both odd, total size $|U| + 1 \geq \Delta(G)$

$\Rightarrow f_o(G) \geq \frac{\Delta(G)}{2}$. ■

Proof ingredients 2

Lemma 2: $\delta(G) \geq 1 \ \Rightarrow f_o(G) \geq \frac{\alpha(G)}{2}$.

Proof: $I \subset V(G)$ – largest independent set, $|I| = \alpha(G)$

$D \subseteq V - I$ – minimal by inclusion set dominating I

(exists as $\delta(G) \geq 1$)

minimality of $D \Rightarrow \forall w \in D \ \exists u_w \in I, N(u_w) \cap D = \{w\}$

(u_w – private neighbor of w)

I_D ≔ set of private neighbors, $|I_D| = D, I_D \subseteq I$

Proof ingredients 2 (cont.)

Choose: $D' \subseteq D$ uniformly at random

$I_0 \subseteq I \setminus I_D$ – vertices with odd degrees into D'

$I_1 = \{ u_w \in I_D : w \in D', w$ has even degree into $D' \cup I_0\}$

$G[I_0 \cup I_1 \cup D']$ – all degrees odd

$$\mathbb{E}[|I_0 \cup I_1 \cup D'|] = \mathbb{E}[|I_0|] + \mathbb{E}[|I_1|] + \mathbb{E}[|D'|] \geq \frac{|I \setminus D|}{2} + \frac{|D|}{2} = \frac{\alpha(G)}{2}.$$

$\Rightarrow \exists$ odd subgraph on $\geq \frac{\alpha(G)}{2}$ vertices. ■

Remark: $\alpha(G) \cdot (\Delta(G) + 1) \geq n \Rightarrow$ recover Caro's estimate $f_o(G) = \Omega(\sqrt{n})$.

Proof ingredients 3

Lemma 3: $G = (V, E)$

M – matching in G with sides U, W

$\forall w \in W$ has only one neighbor in $U \cup W$ (=its mate in M)

[M – semi-induced matching]

Suppose: $|N_G(U) - (W \cup N_G(W))| \geq k$

$\Rightarrow f_o(G) \geq \frac{k}{4}$.

Proof: similar to previous lemmas. ■

Proof idea for the theorem: keep growing such a matching M/parameter k,

or else…

Proof ingredients 4

Lemma 4: $G=(A\cup B,E)$ – bipartite graph, $d(b)>0\ \forall b\in B$

$$\Rightarrow \exists\ (a,b)\in E(G),\ \frac{d(a)}{d(b)}\geq\frac{|B|}{|A|}\ .$$

Proof: (in this formulation – due to Alex Scott)

Choose a random $e=(a,b)\in E(G)$ in two ways:

1. Choose a random $a\in A, d(a)>0$;

   then choose a random $e=(a,b)\in E$; $p_1(e):=$ Pr[e is chosen]$\geq\frac{1}{|A|\cdot d(a)}$

2. Choose a random $b\in B$;

   then choose a random $e=(a,b)\in E$; $p_2(e):=$ Pr[e is chosen] $=\frac{1}{|B|\cdot d(b)}$

Obviously $\sum_e p_1(e)=\sum_e p_2(e)=1\ \Rightarrow\exists\ e, p_1(e)\leq p_2(e)$

For this $e=(a,b)$, $\frac{1}{|A|\cdot d(a)}\leq\frac{1}{|B|\cdot d(b)}$. ■

Key Lemma

Helpful: edge $e = (u, v) \in E(G)$ s.t. $|N(u) \setminus \mathrm{N}(v)| = \Theta(|N(u) \cup N(v)|)$

Then: can add e to matching M from Lemma 3

$\Rightarrow$

<u>Notation:</u>

$$L(G;\beta) = \{v \in V : \exists u \in V, (u, v) \in E(G), |N(u) \setminus N(v)| \geq \beta |N(u) \cup N(v)|\}$$

($\beta > 0$ – small constant)

Large $L(G;\beta) \Rightarrow$ room to operate.

Key Lemma (cont.)

Lemma 5: $G = (V, E), |V| = n, \delta(G) > 0;\ \beta = \frac{1}{20}$

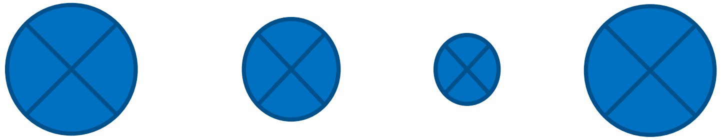
$$|L(G;\beta)| \le \frac{n}{14} \ \Rightarrow f_o(G) \ge \frac{n}{61}\ .$$

Proof: relatively complicated/involved ($\approx$ 2.5 pp)

Main challenge:

Graphs with $|L(G;\beta)|$ small?

Ex.: $G =$ union of disjoint cliques



$L(G;\beta) = \emptyset$

Proof idea: $|L(G;\beta)|$ small $\Rightarrow G \approx$ union of disjoint nearly cliques U_i

$\Rightarrow$ can apply Lemma 1 to each $G[U_i]$,

collect odd pieces from U_i together. ∎

Proof of main theorem 1

Th.: Every graph $G = (V, E), |V| = n, \delta(G) \geq 1,$ contains a subset $V_0 \subseteq V(G), |V_0| \geq \frac{n}{10{,}000}$ s.t. $G[V_0]$ has all degrees odd.

Plan of attack:

Grow a matching M_i with sides U_i, W_i s.t. $|N_G(U_i) \setminus (W_i \cup N_G(W_i))|$ is substantial:

$$\frac{|N_G(U_i) \setminus (W_i \cup N_G(W_i))|}{|N_G(U_i \cup W_i)|} = \Theta(1)$$

If get to: $|N_G(U_i) \setminus (W_i \cup N_G(W_i))| = \Theta(n)$ – can apply Lemma 3, done

Otherwise: look at $V_i = V \setminus N_G(U_i \cup W_i)$

$G[V_i]$: $L(G([V_i]; \beta)$ – small ⇒ apply Key Lemma, done;

$L(G([V_i]; \beta)$ – large ⇒ find an edge e to add to M_i

Proof of main theorem 2

Initialize: $M_0 = \emptyset$

M_i – current matching with sides U_i, W_i

Define: $X_i \coloneqq N(U_i) \setminus (W_i \cup N(W_i))$

Maintain: $\frac{|X_i|}{|N(U_i \cup W_i)|} \geq \frac{1}{40}$.

Can assume: $|X_i| \leq \frac{n}{2{,}500}$ – otherwise done by Lemma 3

$V_i \coloneqq V \setminus N(U_i \cup W_i)$; $|V_i| \geq \frac{n}{2}$.

Look at $G[V_i]$:

$V_i' \coloneqq$ non-isolated vertices in V_i

Can assume: $|V_i'| \geq \frac{n}{4}$ – otherwise large indep. set, done by Lemma 2

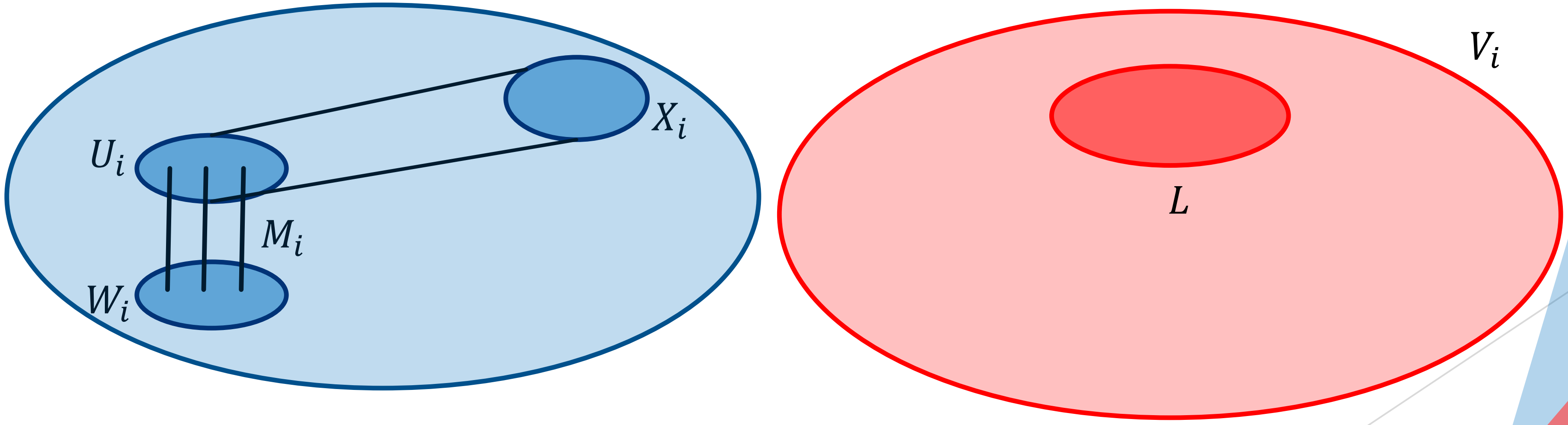
Proof of main theorem 3

Set: $\beta = \frac{1}{20}$

Look at $L \coloneqq L(G[V_i'];\beta)$

$(L(G;\beta) = \{v \in V \colon \exists u \in V, (u,v) \in E(G), |N_G(u) \setminus N_G(v)| \geq \beta |N_G(u) \cup N_G(v)|\})$

Can assume: $|L| \geq \frac{n}{56}$ – otherwise done by Key Lemma



Proof of main theorem 4

Case 1: $\forall v \in L, d(v, X_i) \geq \frac{d(v,V_i)}{40}$

Look at the bipartite graph (X_i, L)

$|X_i| \leq \frac{n}{2{,}500}; |L| \geq \frac{n}{56} \Rightarrow$ apply Lemma 4 to find:

edge $e = (x, v), x \in X_i, v \in L;\ d(x, L) \geq 44\, d(v, X_i) \geq 1.1\, d(v, V_i)$

Then: add e to M_i

Gain to X_i: $\geq d(x, V_i) - d(v, X_i) - d(v, V_i)$

$\geq d(x, V_i)\left(1 - \frac{1}{44} - \frac{10}{11}\right) = \frac{3}{44} d(x, V_i)$

Add to $V \setminus V_i$: $\leq d(x, V_i) + d(v, V_i) \leq d(x, V_i)\left(1 + \frac{10}{11}\right) = \frac{21}{11} d(x, V_i)$

$\Rightarrow$ maintain $\frac{|X_i|}{|V \setminus V_i|} = \Theta(1)$.

Proof of main theorem 5

Case 2: $\exists v \in L, d(v, X_i) \leq \frac{d(v,V_i)}{40}$

$v \in L \Rightarrow \exists\ e = (u, v) \in E(G[V_i']) $ witnessing $v \in L$:

$|N(u, V_i') \setminus N(v, V_i')| \geq \frac{1}{20} |N(\{u, v\}, V_i')|$

Then: add e to M_i

Gain to X_i: $\geq |N(u, V_i') \setminus N(v, V_i')| - |N(v, X_i)|$

$\geq \frac{1}{20} |N(\{u, v\}, V_i')| - \frac{1}{40} |N(v, V_i')| \geq \frac{1}{40} |N(\{u, v\}, V_i')|$

Add to $V \setminus V_i$: $|N(\{u, v\}, V_i')|$

$\Rightarrow$ maintain $\frac{|X_i|}{|V \setminus V_i|} = \Theta(1)$. ∎

Open Problems

- $f_o(n) \geq cn$ (here proved: $c \geq \frac{1}{10{,}000}$)

  Better bounds on c?

  Conditions on graphs G with $f_o(G)$ relatively small?

- Partitioning into induced odd subgraphs?

  – Scott'01

- Other moduli/residues?

  Need: a large subset $V_0 \subseteq V(G)$

  s.t. all degrees in $G[V_0] \equiv i \bmod k$?

  Some results: Caro'94; Scott'01

  random variant ($G \sim G(n, \frac{1}{2})$): Ferber, Hardiman,K.'21+;

  Balister, Powierski, Scott, Tan'21+

Let there be light…

$G = (V, E)$ – graph

- each vertex $v \in V$ has a light and a button
- pressing button at v: switches the light status for v and all its neighbors
- start with all lights off

- Prove: can push some buttons to get all lights on

Solving the riddle:

Looking for a subset $S \subset V$ (= buttons to press) s.t.:

- $\forall v \in S$ has even degree into S;
- $\forall v \in V \setminus S$ has odd degree into S.

➢ Add a new vertex u to V, connect u to all even degree vertices in $G =: G'$

➢ Apply Gallai to G', get two even subgraphs $G'[V_1], G'[V_2]$, assume wlog $u \in V_2$

➢ $S := V_1$ satisfies the required condition.


Odd numbers END in:
1
3
5
7
9