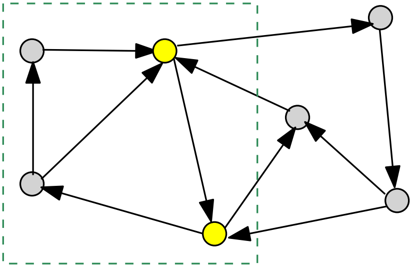
An Optimal Approximation algorithm for FEEDBACK VERTEX SET in Tournaments

Pranabendu Misra

Chennai Mathematical Institute, India.

Feedback Vertex Set

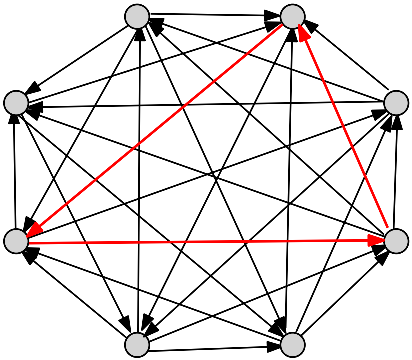
Input: Directed graph G on n vertices.
Output: Minimum subset $S \subseteq V(G)$ such that $G - S$ is acyclic.



It is NP-Complete and has a factor-$O(\log n \log \log n)$ approximation.

Feedback Vertex Set in Tournaments (FVST)

Tournament: A complete digraph, i.e. either (u, v) or (v, u) exists for every pair u, v of vertices.



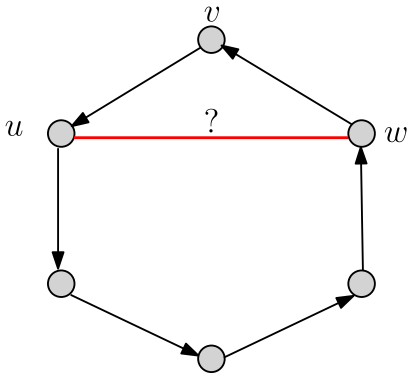
Input: Tournament G on n vertices.
Output: Minimum subset $S \subseteq V(G)$ such that $G - S$ is acyclic.

Still NP-Complete, but easy to get 3-approximation

3-approximation for FVST

Lemma

A tournament G contains a cycle if and only if it contains a cycle of length 3.



Smallest Cycle

3-approximation for FVST

Lemma

A tournament G contains a cycle if and only if it contains a cycle of length 3.

Algorithm: While there exists a triangle $\{u, v, w\}$ in the graph, find and delete all three vertices of it.

Analysis:

- At least one of $\{u, v, w\}$ must be in any optimum solution.
- On deleting $\{u, v, w\}$ the optimum solution decreases by at least 1 vertex, at the cost of 3 vertices.

Better than 3-approximation?

Theorem

Unless the Unique Games Conjecture is false, FVST can't have a factor-$(2-\epsilon)$ approximation algorithm for any constant $\epsilon > 0$.

- [Cai et.al. SICOMP 2000] $5/2$ approximation.

  Local Ratio Technique

- [Mnich et.al. ESA 2016] $7/3$ approximation.

  Iterative Rounding

- [Mathieu, Schudy STOC 2007] PTAS for Feedback Arc Set in tournaments.
- Under Unique Games Conjecture, FVS has no α-approximation for any constant $\alpha > 0$.

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Long standing question: Is there a 2-approximation for FVST?

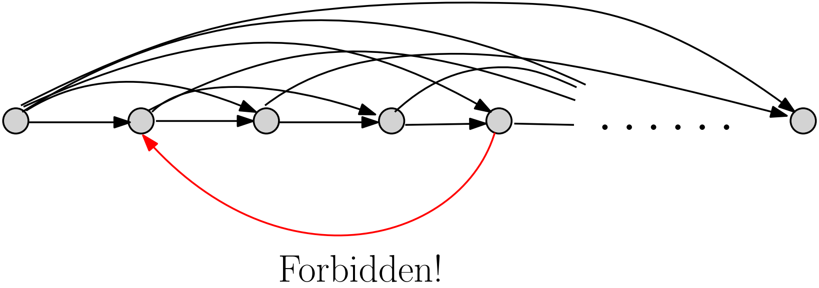
Universal Algorithmic Technique

Pick a random vertex and ... (do something)

Some Simple Ingredients

Lemma

Let G be an acyclic tournament. Then G has a unique topological ordering.



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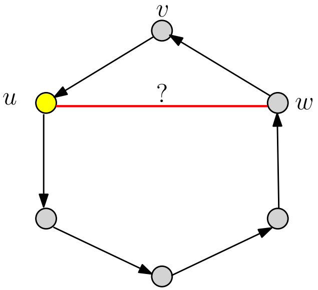
Let G be a tournament, and $u \in V(G)$. Then u is part of a cycle if and only if u is part of a triangle.



Smallest Cycle containing u

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Lemma

Let S_{OPT} denote an optimum solution to FVST in G. If $|S_{OPT}| \geq n/2$ then $V(G)$ itself is a 2-approximate solution.

So we can assume that $|S_{OPT}| < n/2$.

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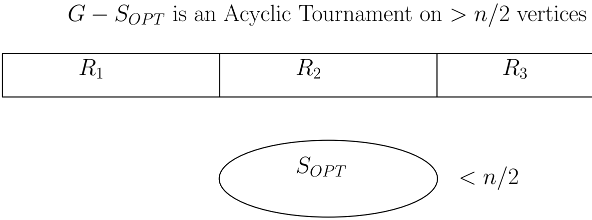
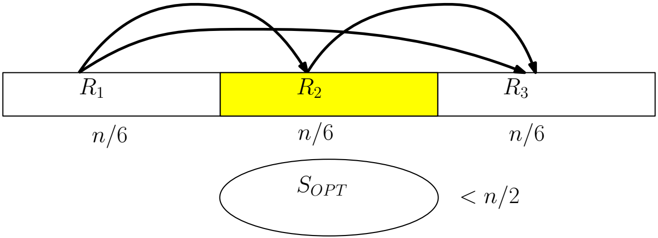
- Let $R = V(G) \setminus S_{OPT}$. Then $G[R]$ is an acyclic tournament on at least $n/2$ vertices.

Note that $G[R]$ is not known to us, but it exists.

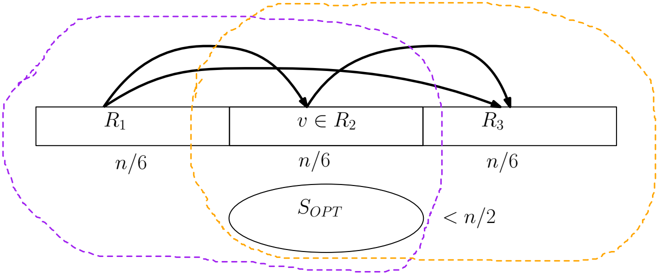
- Consider the unique topological ordering of the vertices of $G[R]$
- Partition R into equal 3 parts, with respect to the topological order, say R_1, R_2, R_3



Any vertex $v \in R_2$ has at least $n/6$ in-neighbors and at least $n/6$ out-neighbors, and there are at least $n/6$ such vertices in G.



Random Sampling



If we pick a vertex $v \in V(G)$ uniformly at random, then with probability at least $\frac{n/6}{n} = \frac{1}{6}$ we have $v \in R_2$.

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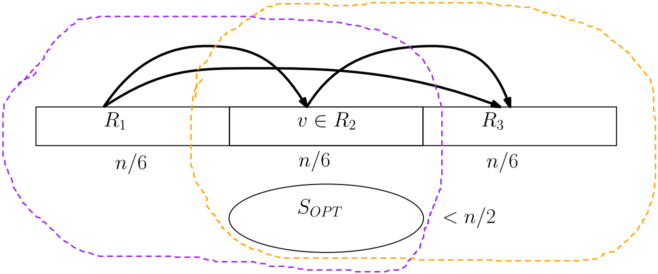

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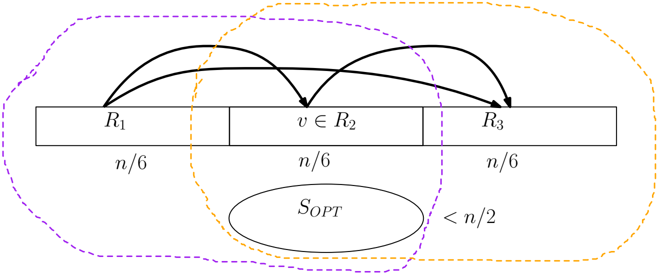

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Key properties of $v \in R_2$:

(1) There is an optimum solution S_{OPT} that excludes v

(2) at most $5n/6$ in-neighbors and $5n/6$ out-neighbors in G.

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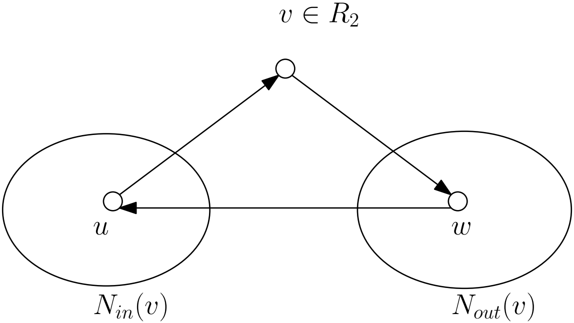
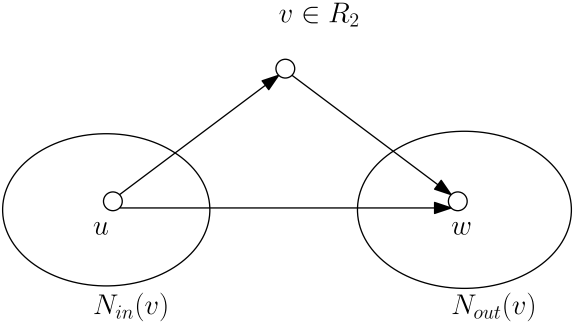


- For every triangle $\{u, v, w\}$ at least one of v, w lies in S_{OPT}.
- Pick v, w into the approximation solution, which is locally 2-approximate.

$$A_v = \text{vertices picked with respect to } v$$

- We stop when there are no more triangles containing v

(2) v has at most $5n/6$ in-neighbors / out-neighbors



- V_1 be the in-neighborhood of v,
  V_2 be the out-neighborhood of v.

$$|V_1|, |V_2| \leq 5n/6$$

- The remaining cycles of G lie in either $G[V_1]$ or $G[V_2]$. Recursively solve these two instances to get A_1, A_2
- Output $A = A_v \cup A_1 \cup A_2$ as the approximate solution.

Approximation factor and Probability

- Pr[$v \in R_2$] $\geq 1/6$
- Let $S_{OPT} = S_v \uplus S_1 \uplus S_2$
  - S_v = all vertices of S_{OPT} that lie in a triangle of v.
  - $S_1 = S \cap V_1$, $|S_1| \geq |S_1^\star|$ an optimum solution of $G[V_1]$.
  - $S_2 = S \cap V_2$, $|S_2| \geq |S_2^\star|$ an optimum solution in $G[V_2]$.
- We know vertices picked for v satisfy $|A_v| \leq 2|S_v|$.
- Suppose the recursive calls on G_1 and G_2 return 2-approximate solutions A_1 and A_2.

$$|A_1| \leq 2|S_1^\star| \leq 2|S_1|, \quad |A_2| \leq 2|S_2^\star| \leq 2|S_2|$$

We assume recursive calls succeed with probability $\geq 1/2$.

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Then,

- $A = A_v \cup A_1 \cup A_2$ is a 2-approximate solution in G
- with probability at least $1/6 * 1/2 * 1/2 = 1/24$

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Boosting the success probability:

Repeat the random sampling 20 times

- For each randomly sampled pivot vertex v_i, we make 2 recursive calls

  we make 40 recursive calls in total

- We obtain a solution A^i for the pivot v_i
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Lemma

If we randomly sample 20 pivot vertices, then with probability at least $1/2$ we obtain a 2-approximation solution.

Running time

- If we sample a vertex $v \in R_2$, then we know that $|V_1|, |V_2| \leq 5n/6$

  The recursive calls are on graphs with at most $5n/6$ vertices
- Using the Master theorem we can bound the running time as:

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Theorem (Lokshtanov et.al. SODA 2020)

There is a randomized polynomial time algorithm for FEEDBACK VERTEX SET *in Tournaments that outputs a* 2*-approximation with probability at least* $1/2$.

Feedback Vertex Set in Tournaments

Our results also extend to the weighted version:

Input : Tournament G, weight function $w : V(G) \rightarrow \mathbb{R}^+$.
Output : Minimum weight subset $S \subseteq V(G)$ such that $G - S$ is acyclic.

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Next questions...

- FVST is an special case of 3-Hitting Set, which cannot have a factor-$3-\epsilon$-approximation algorithm.

  Given universe U and a collection of d-size subsets $\mathcal{F}$, find a minimum $S \subseteq U$ such that $F_i \cap S \neq \emptyset$ for every $F_i \in \mathcal{F}$.

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- (OPEN) Deterministic 2-approximation for FVST?
- (OPEN) Better than 3-approximation for FVS in (subclasses of) Directed Chordal Graphs.

Thank you