An $O(\log \log m)$ Prophet Inequality for Subadditive Combinatorial Auctions

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Joint work with
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Online Combinatorial Auction

- *n* buyers, arriving one by one
- *m* items


- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare

Online Combinatorial Auction

- n buyers, a

$$v_1(\{1\}) = 1$$
$$v_1(\{2\}) = 2$$
$$v_1(\{1,2\}) = 3$$

- m items

- At each arrival: Decide which items to assign (possibly none)
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Online Combinatorial Auction

- n buyers, arriving one b

$$v_2(\{1\}) = 0$$
$$v_2(\{2\}) = 10$$
$$v_2(\{1,2\}) = 10$$

0

- m items

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$$v_2(\{1\}) = 0$$
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- m items

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Online Combinatorial Auction

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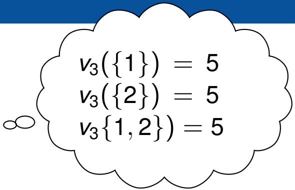

0 10

- m items

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Online Combinatorial Auction

$v_3(\{1\}) = 5$
$v_3(\{2\}) = 5$
$v_3\{1,2\}) = 5$

- n buyers, arriving one by one

0 10 5

- m items

- At each arrival: Decide which items to assign (possibly none)
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Online Combinatorial Auction

- n buyers, arriving one by one

0 10 5

$$v_4(\{1\}) = 20$$
$$v_4(\{2\}) = 50$$
$$v_4(\{1,2\}) = 60$$

- m items

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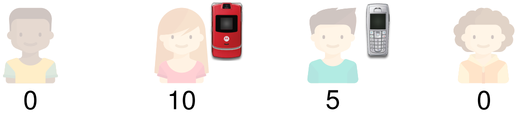
Online Combinatorial Auction

- n buyers, arriving one by one



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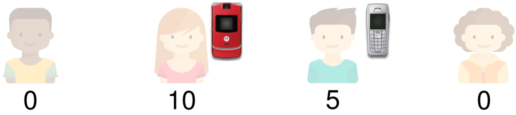

- m items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare
- $v_i \sim \mathcal{D}_i$ independently; $\mathcal{D}_i$ known in advance

Subadditive Valuations

Definition

A valuation function $v_i : 2^{[m]} \to \mathbb{R}_{\geq 0}$ is *subadditive* if

$$v_i(S \cup T) \leq v_i(S) + v_i(T) \quad \text{for all } S, T \subseteq [m]$$

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Definition

A valuation function $v_i : 2^{[m]} \to \mathbb{R}_{\geq 0}$ is *XOS* if

$$v_i(S) = \max_{\ell} \sum_{j \in S} v_{i,j}^{\ell} \quad \text{for all } S \subseteq [m]$$

Prior Work

If all valuation functions are XOS (for example submodular):

- 2-approximation of welfare via static, anonymous item prices (generalizes classic *prophet inequality*) [Feldman, Gravin, Lucier SODA 2015]
- $O(1)$-approximation of revenue via simple mechanism [Cai and Zhao STOC 2017]

Prior Work

If all valuation functions are XOS (for example submodular):

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  via static, anonymous item prices
  (generalizes classic *prophet inequality*)

- $O(1)$-approximation of revenue [Cai and Zhao STOC 2017]
  via simple mechanism

Our question: Valuations are only subadditive (i.e. $v_i(S \cup T) \leq v_i(S) + v_i(T)$)

So far: Only $\Theta(\log m)$-approximations

Our Results

If all valuation functions are subadditive (i.e. $v_i(S \cup T) \leq v_i(S) + v_i(T)$):

- $O(\log \log m)$-approximation of welfare via static, anonymous item prices
- $O(\log \log m)$-approximation of revenue via simple mechanism

Our Results

If all valuation functions are subadditive (i.e. $v_i(S \cup T) \leq v_i(S) + v_i(T)$):

- $O(\log \log m)$-approximation of welfare
  via static, anonymous item prices

- $O(\log \log m)$-approximation of revenue
  via simple mechanism

- Both run in polynomial time given access to demand oracles

Follow-Up Work

- [Assadi, Kesselheim, Singla SODA'21] use our key lemma to design a truthful prior-free $O((\log\log m)^3)$-approximation for XOS and subadditive combinatorial auctions

Outline

1. The balanced prices approach
2. Our new argument
3. Summary and open problems

The Balanced Prices Approach

The Classic Prophet Inequality

Theorem (Samuel-Cahn '84; Kleinberg & Weinberg STOC'12)

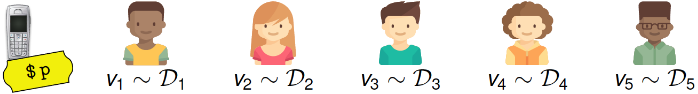
For the single-item problem,

$$\mathbf{E}[ALG(v)] \geq \frac{1}{2} \cdot \mathbf{E}[OPT(v)].$$

Analysis

$v_1 \sim \mathcal{D}_1$ $v_2 \sim \mathcal{D}_2$ $v_3 \sim \mathcal{D}_3$ $v_4 \sim \mathcal{D}_4$ $v_5 \sim \mathcal{D}_5$

Analysis



Set any price p.

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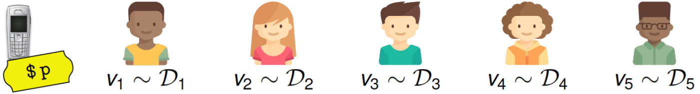
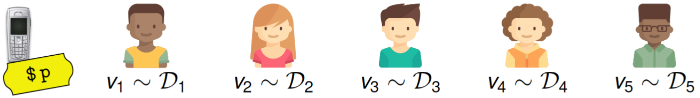


Set any price p. Let $q =$ probability that item is sold.

Analysis



Set any price p. Let q = probability that item is sold.

How much money do we collect?

$$\mathbf{E}[\mathit{revenue}] = p \cdot q$$

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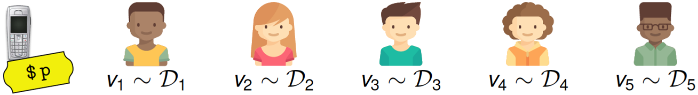

Set any price p. Let $q =$ probability that item is sold.

How much money do we collect?

$$\mathbf{E}[revenue] = p \cdot q$$

What's a buyer's utility (value minus payment)?

$$\begin{aligned} \mathbf{E}[u_i] &= \mathbf{E}[(v_i - p)^+ \cdot \mathbf{1}_{\text{nobody before } i \text{ buys}}] \\ &= \mathbf{E}[(v_i - p)^+] \cdot \mathbf{P}[\text{nobody before } i \text{ buys}] \\ &\geq \mathbf{E}[(v_i - p)^+] \cdot (1 - q) \end{aligned}$$

Putting the Pieces Together

So far:

$$\mathbf{E}[revenue] = p \cdot q \quad \text{and} \quad \mathbf{E}[u_i] \geq \mathbf{E}[(v_i - p)^+] \cdot (1 - q)$$

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In combination:

$$\begin{aligned}\mathbf{E}[\mathit{welfare}] &= \mathbf{E}[\mathit{revenue}] + \sum_i \mathbf{E}[u_i] \\ &\geq p \cdot q + \sum_i \mathbf{E}[(v_i - p)^+] \cdot (1 - q) \\ &\geq p \cdot q + \mathbf{E}[\max_i (v_i - p)] \cdot (1 - q)\end{aligned}$$

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For $p = \frac{1}{2} \cdot \mathbf{E}[\max_i v_i]$ this yields

$$\mathbf{E}[welfare] \geq \frac{1}{2} \cdot \mathbf{E}[\max_i v_i] \cdot q + \frac{1}{2} \cdot \mathbf{E}[\max_i v_i] \cdot (1 - q) = \frac{1}{2} \cdot \mathbf{E}[\max_i v_i]$$

$v_1 = 10$

$v_2 = 30$

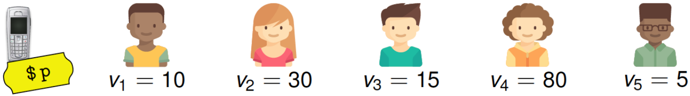
$v_3 = 15$

$v_4 = 80$

$v_5 = 5$

Consider full information.

The Essence



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Price $p = \frac{1}{2} \cdot \max_k v_k$ is “balanced”

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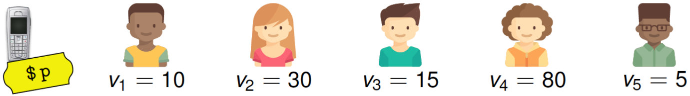

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- **Case 1:** Somebody $i' < i$ buys item

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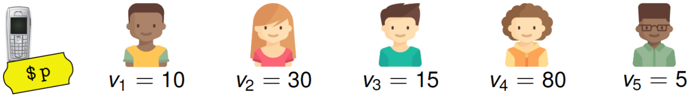

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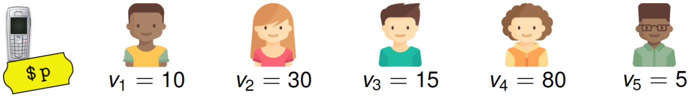

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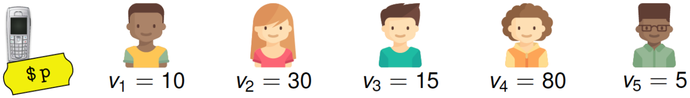

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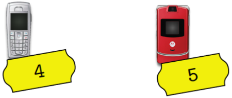
- **Case 1:** Somebody $i' < i$ buys item
  $\Rightarrow$ revenue $\geq \frac{1}{2} v_i$
- **Case 1:** Nobody $i' < i$ buys item
  $\Rightarrow u_i \geq v_i - \frac{1}{2} v_i = \frac{1}{2} v_i$

In either case: welfare $=$ revenue $+$ utilities $\geq \frac{1}{2} v_i$

Posted Prices in Combinatorial Auctions

- n buyers, arriving one by one

- m items



- Precompute item prices $p_1, \ldots, p_m$
- At each arrival: Arriving buyer purchases bundle maximizing utility $v_i(S) - \sum_{j \in S} p_j$
- Maximize social welfare $\sum_{i=1}^{n} v_i(X_i)$

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$$v_1(\{1\}) = 1$$
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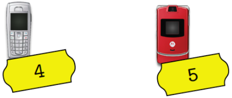

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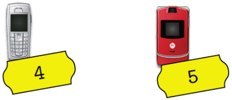

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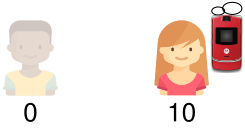

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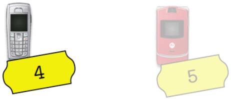

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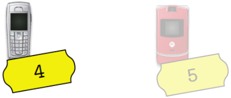

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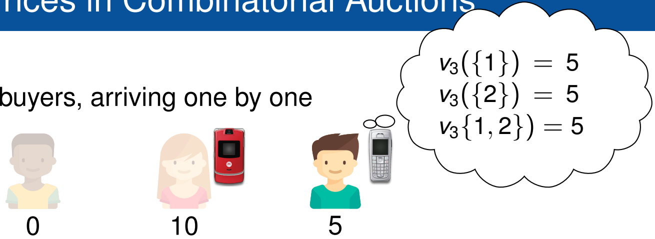

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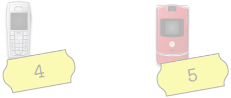

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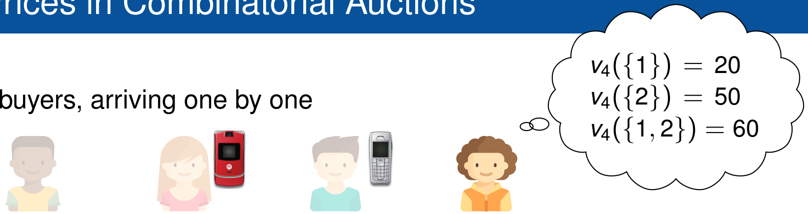

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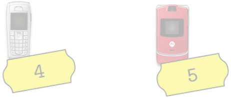

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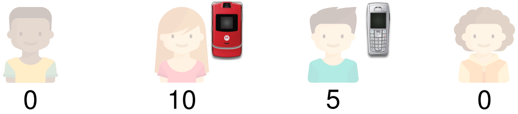

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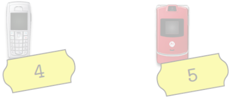

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Prophet Inequality for XOS Combinatorial Auctions

Theorem (Feldman, Gravin, Lucier SODA'15)

For any distributions $\mathcal{D}_1, \ldots, \mathcal{D}_n$ over XOS functions there exist static, anonymous item prices such that for the resulting allocation $X_1, \ldots, X_n$,

$$\mathbf{E}\left[\sum_{i=1}^{n} v_i(X_i)\right] \geq \frac{1}{2} \cdot \mathbf{E}[OPT(v)].$$

Recall: XOS $\Leftrightarrow v_i(S) = \max_\ell \sum_{j \in S} v_{i,j}^\ell$

Balanced Prices: Definition

Definition (Dütting, Feldman, Kesselheim, Lucier FOCS'17)

A valuation function v_i admits balanced prices if for every set of items $U \subseteq [m]$ there exist item prices p_j for $j \in U$ such that for all $T \subseteq U$:

- $\sum_{j \in U \setminus T} p_j \leq v_i(U \setminus T)$ (prices are not too high)
- $\sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T)$ (prices are not too low)

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Observation: XOS functions admit balanced prices

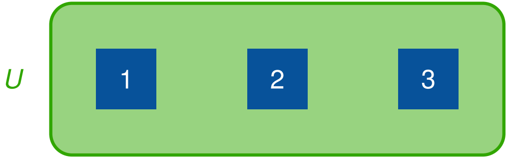
Let ℓ^* be such that $v_i(U) = \sum_{j \in U} v_{i,j}^{\ell^*}$

Let $p_j = v_{i,j}^{\ell^*}$

Balanced Prices: Examples

$$\sum_{j \in U \setminus T} p_j \leq v_i(U \setminus T) \;\; (\forall T \subseteq U) \qquad \sum_{j \in T} p_j \geq v_i(U) - v_i(U \setminus T) \;\; (\forall T \subseteq U)$$

$$U = \{1, 2, 3\}$$



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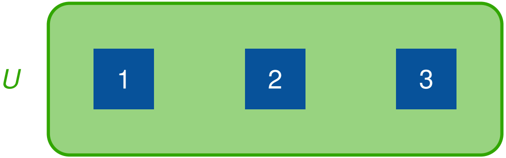

Example 1: Additive

$$v_1(S) = |S|$$

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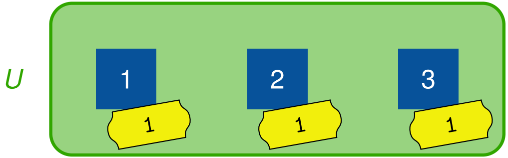

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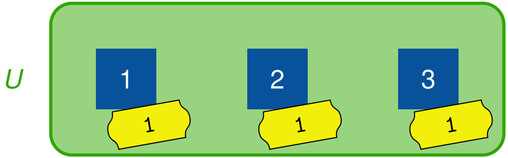

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Balanced Prices: Examples

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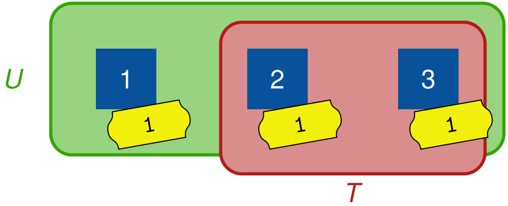

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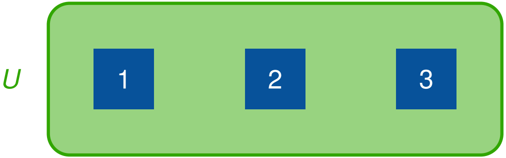

Example 1: Additive

$$v_1(S) = |S|$$

Example 2: Unit-Demand

$$v_2(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ 1 & \text{if } S \neq \emptyset \end{cases}$$

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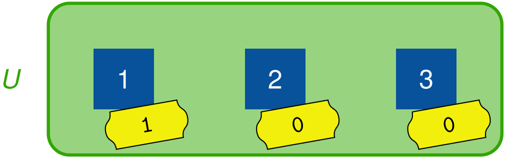

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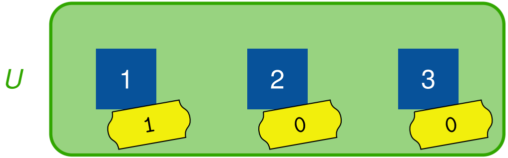

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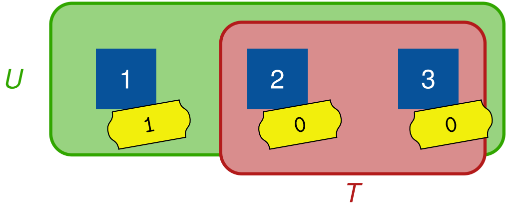

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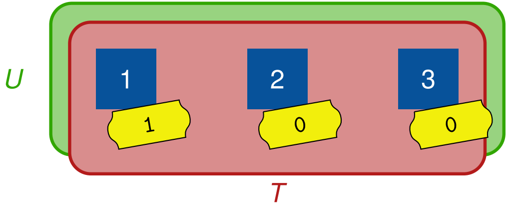

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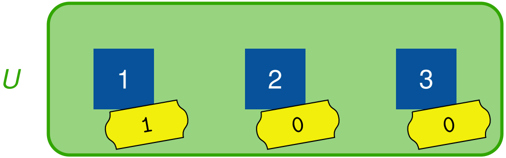

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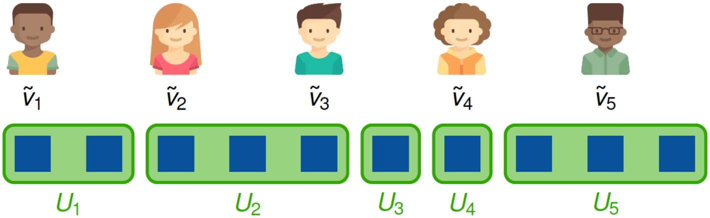
Balanced Prices: General Approximation Bound

Theorem (Dütting, Feldman, Kesselheim, Lucier FOCS'17)

If a class of valuations admits balanced prices, then for any distributions $\mathcal{D}_1, \ldots, \mathcal{D}_n$ there exist static, anonymous item prices such that for the resulting allocation $X_1, \ldots, X_n$,

$$\mathbf{E}\left[\sum_{i=1}^{n} v_i(X_i)\right] \geq \frac{1}{2} \cdot \mathbf{E}[OPT(v)].$$

Setting the Prices



Fix $\tilde{v}_1, \ldots, \tilde{v}_n$

Let $U_i = \{j \mid i \text{ gets } j \text{ in } OPT(\tilde{v})\}$

For $j \in U_i$ set $p_j^{\tilde{v}}$ to balanced price for item j in $\tilde{v}_i$, U_i

Price for item j: $\bar{p}_j = \frac{1}{2} \cdot \mathbf{E}_{\tilde{v} \sim \mathcal{D}}[p_j^{\tilde{v}}]$

Proof Sketch Full Information

Let $U_i = \{j \mid i \text{ gets } j \text{ in } OPT(v)\}$

Set price $\bar{p}_j = \frac{p_j}{2}$ for $j \in U$

Let $T_i = \{j \in U_i \text{ sold to buyers } i' \neq i\}$

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Let $T_i = \{j \in U_i \text{ sold to buyers } i' \neq i\}$

Because prices are balanced:

(a) $\sum_{j \in U_i \setminus T_i} \bar{p}_j \leq \frac{1}{2} v_i(U_i \setminus T_i)$

(b) $\sum_{j \in T_i} \bar{p}_j \geq \frac{1}{2}(v_i(U_i) - v_i(U_i \setminus T_i))$

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Then, for the allocation $X_1, \ldots, X_n$, we have:

$$u_i(X_i, \bar{p}) + \sum_{j \in T_i} \bar{p}_j \geq \left(v_i(U_i \setminus T_i) - \sum_{j \in U_i \setminus T_i} \bar{p}_j \right) + \sum_{j \in T_i} \bar{p}_j$$

$$\geq \left(v_i(U_i \setminus T_i) - \frac{1}{2} v_i(U_i \setminus T_i) \right) + \frac{1}{2} \left(v_i(U_i) - v_i(U_i \setminus T_i) \right)$$

$$= \frac{1}{2} v_i(U_i)$$

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Beyond XOS

- Subadditive functions admit approximately balanced prices
- This way we can get a $\Theta(\log m)$ approximation
- But we cannot do better than this

Our New Argument

Key Lemma

Lemma (Dütting, Kesselheim, Lucier FOCS'20)

For any subadditive valuation v_i and any set $U \subseteq [m]$ there exist prices p_j for $j \in U$ and a probability distribution λ such that for all $T \subseteq U$

$$\sum_{S \subseteq U} \lambda_S \left(v_i(S \setminus T) - \sum_{j \in S \setminus T} p_j \right) + \sum_{j \in T} p_j \geq \frac{v_i(U)}{\gamma},$$

where $\gamma \in O(\log \log m)$.

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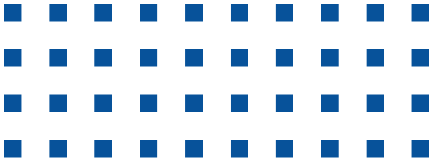

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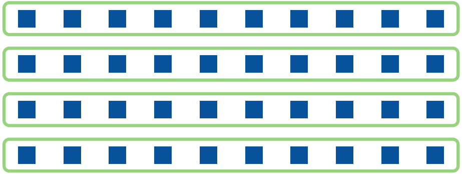

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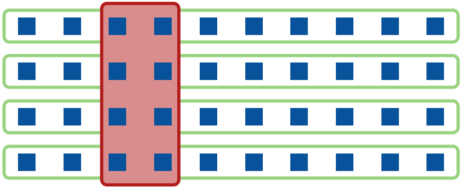

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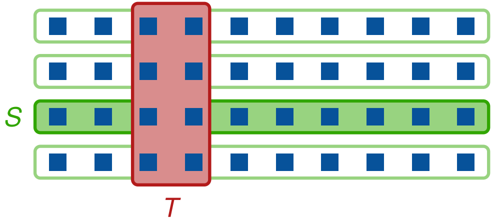

Equivalent to Key Lemma

Lemma

For every subadditive function v_i and set U there exists a probability distribution λ on $S \subseteq U$ so that for every probability distribution μ on $T \subseteq U$ with $\sum_{T: j \in T} \mu_T \leq \sum_{S: j \in S} \lambda_S$ for all items j, it holds that

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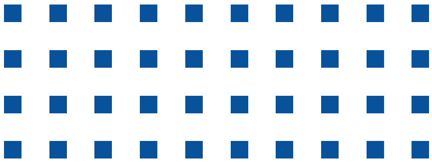

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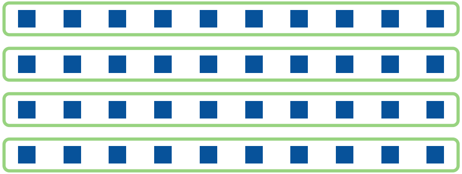

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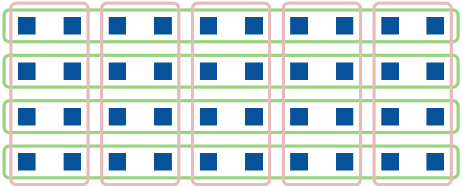

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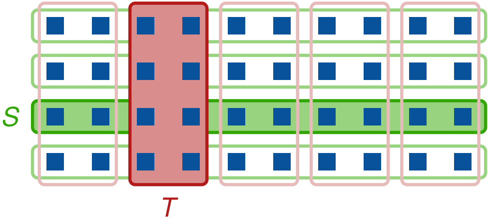

Getting O(log log m)

Claim: There is λ such that for all μ: $\sum_{S,T} \lambda_S \cdot \mu_T \cdot v_i(S \setminus T) \geq \frac{1}{O(\log \log m)} \cdot v_i(U)$.

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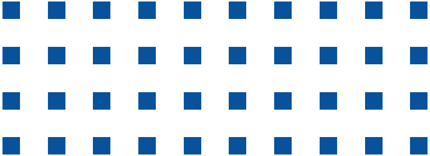

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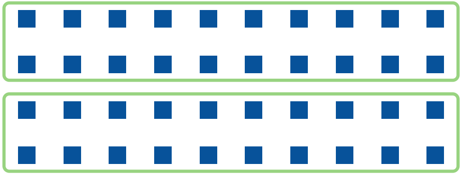

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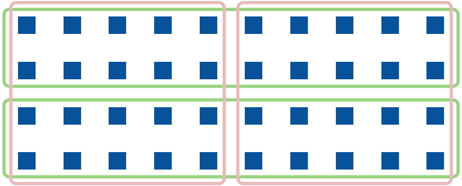

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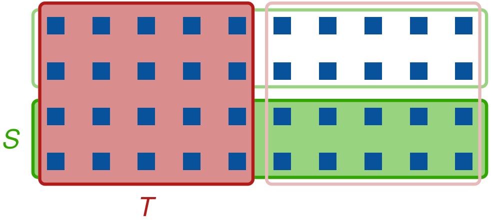

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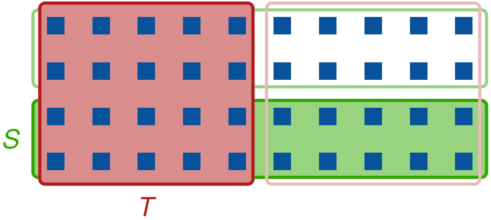
If $\mathbf{E}[v_i(S \setminus T)]$ is small then $\mathbf{E}[v_i(S \cap T)]$ is large.



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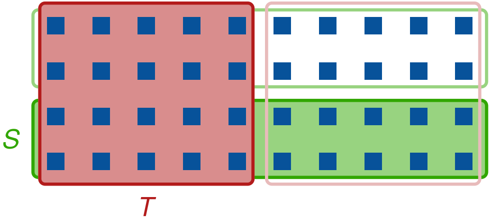
Furthermore: $\mathbf{Pr}[j \in S \cap T] = q^2$.



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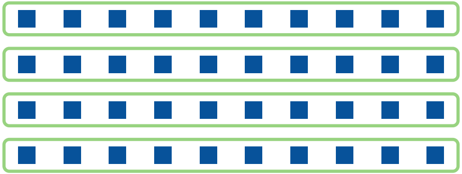

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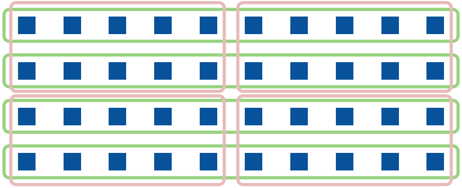

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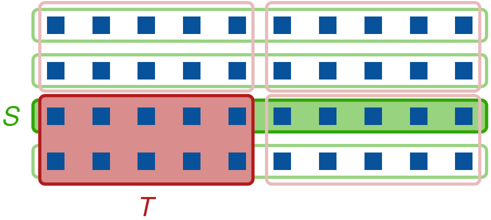
Getting O(log log m)

Claim: There is λ such that for all μ: $\sum_{S,T} \lambda_S \cdot \mu_T \cdot v_i(S \setminus T) \geq \frac{1}{O(\log \log m)} \cdot v_i(U)$.

For $q = \frac{1}{2}, \frac{1}{4}$:

Take λ that maximizes $\sum_S \lambda_S \cdot v_i(S)$ subject to $\sum_{S:j \in S} \lambda_S \leq q$



By subadditivity:
If $\mathbf{E}[v_i(S \setminus T)]$ is small then $\mathbf{E}[v_i(S \cap T)]$ is large.

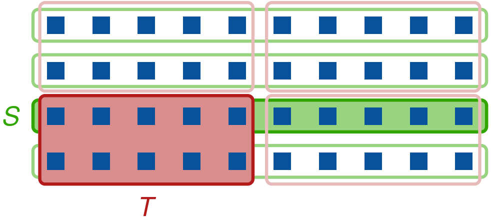
Furthermore: $\mathbf{Pr}[j \in S \cap T] = q^2$.

Getting O(log log m)

Claim: There is λ such that for all μ: $\sum_{S,T} \lambda_S \cdot \mu_T \cdot v_i(S \setminus T) \geq \frac{1}{O(\log\log m)} \cdot v_i(U)$.

For $q = \frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \ldots, \frac{1}{m}$:

Take λ that maximizes $\sum_S \lambda_S \cdot v_i(S)$
subject to $\sum_{S:j \in S} \lambda_S \leq q$



By subadditivity:
If $\mathbf{E}[v_i(S \setminus T)]$ is small then $\mathbf{E}[v_i(S \cap T)]$ is large.

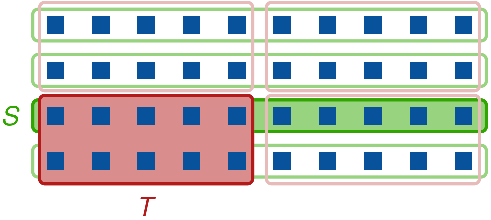
Furthermore: $\mathbf{Pr}[j \in S \cap T] = q^2$.

Getting O(log log m)

Claim: There is λ such that for all μ: $\sum_{S,T} \lambda_S \cdot \mu_T \cdot v_i(S \setminus T) \geq \frac{1}{O(\log \log m)} \cdot v_i(U)$.

For $q = \frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \ldots, \frac{1}{m}$:

Take λ that maximizes $\sum_S \lambda_S \cdot v_i(S)$
subject to $\sum_{S: j \in S} \lambda_S \leq q$



By subadditivity:
If $\mathbf{E}[v_i(S \setminus T)]$ is small then $\mathbf{E}[v_i(S \cap T)]$ is large.

Furthermore: $\mathbf{Pr}[j \in S \cap T] = q^2$.

$\Rightarrow$ One of $q = \frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \ldots, \frac{1}{m}$ will be good.

Additional Results in the Paper

Additional Results

- The $O(\log\log m)$ bound is tight for the equal marginals approach taken here
- An alternative proof of key lemma based on configuration LP, which yields an efficient algorithm
- A simple, DSIC mechanism that yields a $O(\log\log m)$ approximation to the optimal revenue

Conclusion and Open Questions

Summary

- Major progress on one of the main frontiers in the posted pricing/ prophet inequalities literature
- Technique for dealing with subadditive valuations that goes beyond "approximate with XOS functions"
- Big open question: Can we get $O(1)$?

Thanks! Questions?